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SOLVABILITY OF THE FREE BOUNDARY VALUE PROBLEM OF THE NAVIER-STOKES EQUATIONS

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Abstract. In this paper, we study the incompressible Navier-Stokes equations on a moving domain in $\mathbb{R}^3$ of finite depth, bounded above by the free surface and bounded below by a solid flat bottom. We prove that there exists a unique, global-in-time solution to the problem provided that the initial velocity field and the initial profile of the boundary are sufficiently small in Sobolev spaces.

1. **Introduction.** In this paper, we study a viscous free boundary value problem with surface tension. The Navier-Stokes equations describe the evolution of the velocity field in the fluid body. With boundary conditions stated below, we have the following system of equations:

$$(NSF)\begin{cases} v_t + v\cdot\nabla v - \mu\Delta v + \nabla p = 0 \quad \text{in} \quad \Omega_t, \\ \nabla\cdot v = 0 \quad \text{in} \quad \Omega_t, \\ v = 0 \quad \text{on} \quad S_B, \\ \eta_t = v_3 - v_1\partial_x\eta - v_2\partial_y\eta \quad \text{on} \quad S_F, \\ pn_i = \mu(v_{i,j}+v_{j,i})n_j + \Big(g\eta - \beta\nabla\cdot(\dfrac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}})\Big)n_i \quad \text{on} \quad S_F, \end{cases}$$

where $\Omega_t = \{(x,y,z) : -1 < z < \eta(x,y,t)\}$ having two boundaries $S_F = \{(x,y,z) : z = \eta(x,y,t)\}$ and $S_B = \{(x,y,z) : z = -1\}$. $\hat{n} = (n_1,n_2,n_3)$ is the outward normal vector on S_F. μ is the constant of viscosity, g is the gravitational constant, and β is the constant of surface tension. From now on, we normalize all the constants by 1. (We follow the Einstein convention where we sum upon repeated indices. Subscripts after commas denote derivatives.)

The boundary condition of the velocity at the bottom S_B is the Dirichlet condition, $v = 0$, which is the boundary condition of the Navier-Stokes equations on a fixed domain. Therefore, we can apply the Poincare inequality to control lower order terms by using higher order terms.

On the free surface $S_F = \{(x,y,z); z = \eta(x,y,t)\}$, we have three boundary conditions:

• the kinematic condition: we represent the free boundary by $d(x,y,z,t) = z - \eta(x,y,t) = 0$. Since the free boundary moves with the fluid, $(\partial_t + v\cdot\nabla)(z -$

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$\eta(x,y,t)) = 0$, from which $\eta_t = v_3 - v_1\partial_x\eta - v_2\partial_y\eta$.

- the shear stress boundary condition: $(\hat{t}\cdot\nabla v\cdot\hat{n} + \hat{n}\cdot\nabla v\cdot\hat{t}) = 0$, where $\hat{t}$ is any tangential vector on the free boundary and $\hat{n} = \frac{1}{\sqrt{1+|\nabla\eta|^2}}(-\partial_x\eta, -\partial_y\eta, 1)$.
- the normal force balance condition: $pn_i = (v_{i,j}+v_{j,i})n_j + \eta n_i - \nabla\cdot(\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}})n_i$.

Since the problem is posed on a domain, compatibility conditions for the initial data are needed and are as follows:

$$\begin{cases} \{(v_0)_{i,j} + (v_0)_{j,i}\}_{\tan} = 0 \quad \text{on} \quad S_F = \{(x,y,z) : z = \eta_0(x,y)\}, \\ \nabla\cdot v_0 = 0 \quad \text{in} \quad \Omega_0, \\ v_0 = 0 \quad \text{on} \quad S_B = \{(x,y,z) : z = -b\}, \end{cases}$$

where (tan) is the tangential component, and the first condition is obtained by taking the inner product with the pressure on the initial surface and any tangential vector.

Let us briefly compare the free boundary problem of the Euler equations with that of the Navier-Stokes equations. Earlier works on the free boundary problem of the Euler equations were treated under the assumption that the flow is ir-rotational. The fluid motion is described by a velocity potential which is harmonic, and such a system can be reduced into a system where all the functions are projected on the free surface. See [8] for the system of equations on the free surface. The first break through in solving the well posedness for the ir-rotational Euler equations without surface tension, for general data was attributed to Wu [20], [21]. However, for the Navier-Stokes equations, it is impossible to assume that the flow is ir-rotational from the following reason. The shear stress condition implies that the tangential part of the vorticity on the boundary satisfies

$$w_T = w - (w\cdot\hat{n})\hat{n} = -2\hat{n}\times\nabla v\cdot\hat{n} = -2(\hat{n}\times\nabla)\cdot(\hat{n}\cdot v) + 2u_j((\hat{n}\times\nabla)n_j),$$

where $(\hat{n}\times\nabla) = (n_2\partial_z - n_3\partial_y, n_3\partial_x - n_1\partial_z, n_1\partial_y - n_2\partial_x)$ is a tangential derivative. This condition prevents a viscous flow from being ir-rotational as is evident in two dimensional flow. In a local coordinate system, the vorticity w at the free surface is given by $w = \hat{n}\cdot\nabla v\cdot\hat{t} - \hat{t}\cdot\nabla v\cdot\hat{n}$. From the shear stress condition, we rewrite w as

$$w = -2\hat{t}\cdot\nabla v\cdot\hat{n} = -2\frac{\partial v}{\partial s}\cdot\hat{v} = -2\frac{\partial}{\partial s}(v\cdot\hat{n}) + 2u\cdot\frac{\partial\hat{n}}{\partial s} = -2\frac{\partial}{\partial s}(v\cdot\hat{n}) + 2(v\cdot\hat{t})\kappa,$$

where κ is the curvature of the surface. This means that the vorticity develops at the free surface whenever there is relative flow along a curved surface so that the vorticity does not vanish at the free surface. See [12]. For the recent works on the Euler equations for the rotational case, see [2, 7, 11, 14, 22]. In particular, in [14] their approach is based on the geometric interpretation of the Euler equations as a flow in the space of volume preserving maps and on the variational formulation of the free boundary problems. We will use a similar idea used in [14] to obtain the a priori estimate in section 2.

The second difference highlighted in this paper is the instability condition. One of the main issues for the Euler equations is the Rayleigh-Taylor sign condition of the pressure term. In the absence of this condition, Ebin [10] proved that the problem is ill-posed. In the presence of surface tension, the pressure term becomes a lower order term so that instability does not occur. The role of surface tension related to the Rayleigh-Taylor instability and its regularizing effect is well explained in [14] in terms of differential operators defined by identifying the correct linearized problem. For the Navier-Stokes equations, however, the pressure term is a lower order term even with surface tension. Moreover, the viscosity alone provides all the necessary regularizing effects on the velocity field. Surface tension plays a different role for the Navier-Stokes equations. It provides higher regularity for the boundary function and generates more decays on the boundary function as well so that we can obtain a global-in-time result.

Before we proceed to the results in this paper, let us present some existing results of the Navier-Stokes equations on a moving domain. In the presence of surface tension, in [4], Beale studied the motion of a viscous incompressible fluid contained in a three dimensional ocean of infinite extent, bounded below by a solid floor and above by an atmosphere of constant pressure. His approach is to transform the problem to the equilibrium domain dependent on the unknown η. The entire problem can be solved by iteration in K^r. (For definition of parabolic-type Sobolev spaces K^r, see [4].) In the absence of surface tension, Beale, in [3], showed the local well-posedness for arbitrary initial data with certain regularity assumptions. He also proved that for any fixed time interval, solutions exist provided the initial data are sufficiently close to the equilibrium. Along the same lines in [4], Sylvester [13] showed that viscosity alone prevents the formation of singularities so that solutions exist globally-in-time for small initial data with higher regularity. In the case of a bounded domain, Solonnikov investigated the fluid motion of a fluid of finite mass, and obtained a global-in-time solution with [16] or without [17] surface tension. In a three dimensional domain of infinite extent and finite depth, Tani-Tanaka [18] solved the problem with or without surface tension using Solonnikov's method rather than Beale's method. These three works [16, 17, 18] were also dealt in K^r space. In a bounded domain, Coutand-Shkoller [6] used energy methods to establish the a priori estimate which allows to find a unique weak solution to the linearized problem in the Lagrangian coordinates, then applied the topological fixed point theorem to obtain a solution. In [6], they obtained the a priori estimate in spaces which are almost the same space for the Navier-Stokes equations on a fixed domain. (We will explain in more details the results [4] and [6] below.)

The paper is organized according to the following outline. In section 2 we will obtain the a priori estimate on the moving domain. The basic L^2 energy estimate is easily derived by multiplying the momentum equation by v and integrating over the spatial variables. If the problem is posed on the whole space, we can obtain higher energy bounds by taking derivatives of the equations, while we cannot take usual partial derivatives to equations on the moving domain because the domain is not translation-invariant in the spatial variables. Instead, we will obtain global-in-time estimates on the moving domain using a second order differential operator, which is derived by projecting the momentum equation onto the divergence-free space. However, we do not know how to iterate the system locally-in-time on the moving domain so that we cannot solve the problem by obtaining the a priori estimate on the moving domain first. But, the importance of these estimates is that most of calculations used in the following sections are based on these estimates. Moreover, we can choose the regularity of initial data from the new formulation of the momentum equations in section 2. Finally, as we know, this is the first result of obtaining the a priori estimate on the moving domain without transforming the system of equations to a fixed domain.

In this paper, we solve the problem by fixing the domain first, and then deal with the problem on the fixed domain. By reversing steps, we can solve the problem on the original moving domain. Traditionally, one might fix the domain by the Lagrangian map. Then, the solvability of the problem is strongly dependent on the L^1 in-time estimate of the velocity. But, we have L^∞ or L^2 in-time estimates of the velocity when we use the usual energy estimates for the Navier-Stokes equations so that we only expect local-in-time results if we fix the domain by using the Lagrangian map. As an example, we present the work of Coutand-Shkoller [6]. Let $\Omega_0 \subset R^3$ denote an open bounded domain with boundary $\Gamma_0 = \partial\Omega_0$. For each $t \in (0,T]$, we wish to find the domain Ω_t, a divergence-free velocity field $u(t)$, a pressure $p(t)$, and a volume preserving transformation $\eta(t) : \Omega_0 \to R^3$ such that

$$\begin{cases} \Omega_t = \eta(t,\Omega_0), \quad \partial_t\eta(t,x) = u(t,\eta(t,x)), \\ u_t - \Delta u + (u\cdot\nabla)u\nabla v + \nabla p = f, \\ \nabla\cdot u = 0, \\ (Defu)\cdot\hat{n} - p\hat{n} = \sigma H\hat{n} \quad \text{on} \quad \Gamma_t, \end{cases}$$

where σ denotes surface tension, H denotes the mean curvature of the surface, and $Defu$ is twice the rate of deformation tensor of u. Let $a(x) = (\nabla\eta)^{-1}$, $v = u\circ\eta$ denote the Lagrangian velocity field, $q = p\circ\eta$ is the Lagrangian pressure, and $F = f\circ\eta$ is the forcing function. Then, the above system can be written as

$$\begin{cases} \partial_t\eta = v, \quad v^i_t - (a^j_l a^k_l v^i_{,k})_{,j} + a^k_i q_{,k} = F^i \partial_t\eta(t,x) = u(t,\eta(t,x)), \\ (v^i_{,k}a^k_l + v^l_{,k}a^k_i)a^j_l N_j - q a^j_i N_j = \sigma\Delta_g(\eta)^i, \\ a^k_i v^i_{,k} = 0, \end{cases}$$

where N denotes the outward unit normal to Γ_0 and $\Delta_g(\eta) = (H\hat{n})\circ\eta$, and they prove the following theorem.

Theorem 1.1. *Let $\Omega_0 \subset R^3$ be a smooth, open and bounded subset, and suppose $u_0 \in H^2$ satisfies the compatibility condition $[Defu_0N]_{tan} = 0$ and that $f \in L^2(0,T;H^1)$, $f_t \in L^2(0,T;(H^1)')$. Then, there exists a $T > 0$ such that there exists a solution to the problem. Furthermore, $\eta \in C_0([0,T];H^3)$, and $\sigma\Delta_g(\eta) \in L^2(0,T;H^{\frac{3}{2}}(\Gamma_0))$. Moreover, the solution is unique if f, f_t, and ∇f are uniformly Lipschitz in the spatial variables.*

For the Navier-Stokes equations on the whole space, however, we can obtain a global-in-time solution for small initial data. This is the first motivation of this paper. Namely, we want to obtain a global-in-time result for small initial data even under the influence of the moving surface. In order to obtain a global-in-time result, we will solve the problem on the equilibrium domain. The transformation from the moving domain to the equilibrium domain will be presented in section 3. The transformed system of equations on the equilibrium domain is given by

$$(LNSF)\begin{cases} w_t - \Delta w + \nabla q = f \quad \text{in} \quad \Omega = \{(x,y,z) : -1 < z < 0\}, \\ \nabla\cdot w = 0 \quad \text{in} \quad \Omega, \\ w_{i,3} + w_{3,i} = g_i \quad \text{on} \quad \{z=0\}, \quad \eta_t = w_3 \quad \text{on} \quad \{z=0\}, \\ q = w_{3,3} + \eta - \Delta_0\eta + g_3 \quad \text{on} \quad \{z=0\}, \\ w = 0 \quad \text{on} \quad \{z=-1\}, \end{cases}$$

where f and g_i are quadratic functions of w and η. If we solve this linearized problem, then we can solve the full problem on the equilibrium domain by the contraction mapping theorem. This idea can be found in [3, 4]. Here, we present the main result in [4].

Theorem 1.2. *Suppose r is chosen with $3 < r < \frac{7}{2}$. There exists $\delta > 0$ such that for v_0 and η_0 satisfying $\|\eta_0\|_{H^r(R^2)} + \|v_0\|_{H^{r-\frac{1}{2}}(\Omega_0)} \le \delta$ and the compatibility conditions, the problem has a solution v, η and p, where $\eta \in \tilde{K}^{r+\frac{1}{2}}(R^2 \times R^+)$ and v and p are restrictions to the fluid domain Ω_t of functions defined on $R^3 \times R^+$, with $v \in K^r(R^3 \times R^+)$ and $\nabla p \in K^{r-2}(R^3 \times R^+)$.*

Let $r = 3 + \delta$. $v \in K^r$ implies that $v \in H_t^{\frac{s}{2}} H_x^{r-s}$, which is embedded in $C_t H_x^{r-s}$ if $s > 1$. By setting $s = 1 + \epsilon$, $r - s = 2 + \delta - \epsilon$. Since the initial data is in $H^{r-\frac{1}{2}}$ and $r - \frac{1}{2} > 2 + \delta - \epsilon$, the solution does not preserve the initial regularity $H^{r-\frac{1}{2}}$ as it evolves in-time. This happens because he solved the problem by taking the Laplace transform in-time to make the system of equations stationary. Let λ be a dual variable of the time variable. Then, the momentum equation of the velocity field and the evolution equation of the boundary become

$$\lambda \hat{\eta} = R\hat{w}, \quad \lambda \hat{w} + A\hat{w} + E(1 - \Delta)\hat{\eta} = \hat{f},$$

where A is a positive definite, self-adjoint operator, E is the formal adjoint with L^2 norm of the restriction operator R on the free boundary. By substituting the first equation into the second,

$$\lambda \hat{w} + A\hat{w} + \frac{1}{\lambda} B\hat{w} = \hat{f},$$

where $B = E(1 - \Delta)R$. He obtained the a priori estimate of this equation by considering large λ and small λ separately, which implies that in the original time variable, the solution is in L^2 in-time, not L^∞ in-time. This is the second motivation of the paper. Our goal is to obtain a solution in L^∞ in-time.

Following [4], we will solve the problem on the equilibrium domain, but without taking the Laplace transform in-time to transform the time evolution problem into the stationary elliptic problem. Instead, we will obtain a solution by using the energy method in the same space used for the Navier-Stokes equations on a fixed domain. In section 3 we present how we solve the problem on the equilibrium domain under the assumption of the solvability of (LNSF). In section 4 we will prove that (LNSF) has a weak solution in L^2, and it has higher regularity under higher regularity of initial data and external forces (Proposition 4.1). To prove Proposition 4.1, from the face that the domain is translation-invariant in the horizontal direction, we first take tangential derivatives to the momentum equation to obtain energy bounds of tangential derivatives of the velocity field. Other bounds can be derived from the divergence-free condition and from the momentum equation. However, we cannot obtain the L^2 in-time estimates of the boundary directly from Proposition 4.1. But, we can deduce those L^2 in-time estimates by projecting the momentum equation onto the divergence-free space and following the arguments in section 2. Having solved the linearized problem, we reverse our steps and obtain a solution of the original problem. In section 6, we present proofs of results in section 2. The main result of this paper is the following.

Theorem 1.3. *Suppose $v_0 \in H^2$ and $\eta_0 \in H^3$. If initial data are sufficiently small, then there is a unique, global-in-time solution (v, η, p) to (NSF) such that*

$$|||(v, \eta, p)||| \lesssim \|v_0\|_{H^2} + \|\eta_0\|_{H^3},$$

where

$$|||(v, \eta, p)||| = \|v\|_{L_t^\infty H_x^2} + \|v\|_{L_t^2 H_x^3} + \|\eta\|_{L_t^\infty H_x^3} + \|\nabla \mathscr{H}(\eta - F(\eta))\|_{L_t^2 H_x^1} + \|\nabla p\|_{L_t^2 H_x^1}.$$

Notations: • $(f,g) = \int fg dV$, • $\epsilon > 0$ is the size of the initial data.

• $< u, u >= \frac{1}{2}\int_\Omega (u_{i,j}+u_{j,i})(u_{i,j}+u_{j,i})dv, \quad F(\eta) = \nabla\cdot(\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}})$.

• $\mathscr{H}$ is the harmonic extension operator, also denoted by $\mathscr{H}(f) = \bar{f}$, extending functions defined on S_F to $\dot{H}^1$ harmonic functions on Ω with zero Neumann boundary condition on S_B.

• $A \lesssim B$ means there is a constant C such that $A \le CB$. $A \lesssim B + \frac{1}{2}D$ means there is a constant C' such that $A \le C' B + \frac{1}{2}D$.

• ∇_0 is a tangential derivative along the $x-y$ plane.

• $\hat{n}\cdot T_v \cdot \hat{n} = \sum_{i,j} n_i(v_{i,j}+v_{j,i})n_j$.

2. **A priori estimate on the moving domain.** In this section we will establish the a priori estimate on the moving domain. The basic L^2 estimate can be easily obtained by multiplying the momentum equation by v and integrating the equation in the spatial variables. Since we cannot take usual partial derivatives to the equations on the moving domain, we need to take a special differential operator which is derived from the new expression of the momentum equation. Here, we only present a sketch of the arguments. For details, see section 6.

Theorem 2.1. *Let $v_0 \in H^2$ and $\eta_0 \in H^3$. If initial data are sufficiently small, then a global-in-time solution (v,η,p) satisfies*

$$|||(v,\eta,p)||| \lesssim \|v_0\|_{H^2} + \|\eta_0\|_{H^3} + (|||(v,\eta,p)|||)^2.$$

2.1. **Basic energy estimate.** We apply the energy method to (NSF) in the physical domain. We multiply the momentum equation by v and integrate over Ω_t.

$$\begin{aligned}0 =& \int_{\Omega_t}\frac{1}{2}\frac{d}{dt}|v|^2dV + \int_{\Omega_t}\frac{1}{2}\nabla\cdot(v|v|^2)dV - \int_{\Omega_t}(\Delta v)\cdot v dV + \int_{\Omega_t}\nabla p\cdot v dV\\ =&\frac{1}{2}\frac{d}{dt}\int_{\Omega_t}|v|^2dV - \frac{1}{2}\int_{\partial\Omega_t}(v\cdot\hat{n})|v|^2dS + \frac{1}{2}\int_{\partial\Omega_t}(v\cdot\hat{n})|v|^2dS\\ &+\frac{1}{2}\int_{\Omega_t}|v_{i,j}+v_{j,i}|^2dV - \int_{\partial\Omega_t}(v_{i,j}+v_{j,i})n_jv_idS + \int_{\partial\Omega_t}pn_iv_idS.\end{aligned} \tag{2.1}$$

From the boundary condition of the pressure on the free surface, we obtain that

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_t}|v|^2dV + \frac{1}{2}\int_{\Omega_t}|v_{i,j}+v_{j,i}|^2dV + \int_{\partial\Omega_t}(v\cdot\hat{n})(\eta - F(\eta))dS = 0. \tag{2.2}$$

Since $(v\cdot\hat{n}) = \frac{\eta_t}{\sqrt{1+|\nabla\eta|^2}}$,

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_t}|v|^2dV + \frac{1}{2}\int_{\Omega_t}|v_{i,j}+v_{j,i}|^2dV + \int_{\partial\Omega_t}\frac{\eta_t}{\sqrt{1+|\nabla\eta|^2}}(\eta - F(\eta))dS = 0. \tag{2.3}$$

By the change of variables, we can replace the last term in (2.3) by

$$\begin{aligned}&\int_{\partial\Omega_t}\frac{\eta_t}{\sqrt{1+|\nabla\eta|^2}}(\eta - F(\eta))dS = \frac{1}{2}\frac{d}{dt}\int|\eta|^2 + (\sqrt{1+|\nabla\eta|^2}-1)dxdy\\ &= \frac{1}{2}\frac{d}{dt}\int|\eta|^2 + \frac{|\nabla\eta|^2}{1+\sqrt{1+|\nabla\eta|^2}}dxdy,\end{aligned} \tag{2.4}$$

from which we can rewrite the equation (2.2) as

$$\frac{d}{dt}\int_{\Omega_t}|v|^2dV+\int_{\Omega_t}|v_{i,j}+v_{j,i}|^2dV+\frac{d}{dt}\int|\eta|^2+\frac{|\nabla\eta|^2}{1+\sqrt{1+|\nabla\eta|^2}}dxdy=0. \quad (2.5)$$

We integrate the equation (2.5) in-time. By Korn's inequality (Lemma 6.6), we replace the symmetric part of the gradient of velocity field by the full derivative.

$$\|v(t)\|_{L^2}^2+\|\eta(t)\|_{L^2}^2+\int\frac{|\nabla\eta|^2}{1+\sqrt{1+|\nabla\eta|^2}}dxdy+\int_0^t\int_{\Omega_s}|\nabla v|^2dVds\lesssim\epsilon. \quad (2.6)$$

To conclude the basic L^2 estimate, we need to show that $\|\nabla\eta\|_{L^\infty_x}$ is uniformed bounded for all time, and this will be established by higher energy estimates. Under this boundedness of η, we can obtain the basic L^2 bound:

$$\|v\|^2_{L^\infty_tL^2_x}+\|\nabla v\|^2_{L^2_tL^2_x}+\|\eta\|^2_{L^\infty_tL^2_x}+\|\nabla\eta\|^2_{L^\infty_tL^2_x}\lesssim\epsilon. \quad (2.7)$$

2.2. **New formulation of the momentum equation.** Now, we use the vector field decomposition method to rewrite the momentum equation in such a way that the pressure in the fluid body can be expressed as the harmonic extension of the pressure on the boundary by projecting the momentum equation onto the divergence-free space. A similar projection has been used in treating the incompressible Navier-Stokes equations on a fixed domain, where the pressure term has no effect on the projected equation. But, on a moving domain, parts of the pressure on the boundary still remain in the projected equation.

To this end, let us start with the Hodge decomposition. Any vector field X in Ω can be written as a sum of a divergence-free vector field and a gradient: $X=u+\nabla\phi$. From the identity

$$\int_\Omega u\cdot\nabla\phi dV+\int_\Omega(\nabla\cdot u)\phi dV=\int_{\partial\Omega}(u\cdot\hat{n})\phi,$$

we conclude that u is of divergence-free and $u\cdot\hat{n}=0$ on S_B is L^2 orthogonal to $\nabla\phi$ with $\phi=0$ on S_F. We denote u by $\mathbb{P}X$. Here, we list two properties of the operator $\mathbb{P}$. For the proof, see [3].

Lemma 2.2. *(1) It is a bounded operator on H^s.*
(2) If $\phi\in H^1$, then $\mathbb{P}(\nabla\phi)=\nabla\mathscr{H}(\pi)$, where $\phi=\pi$ on S_F.

In our problem, the velocity field v and its time derivative v_t are in the range of $\mathbb{P}$. Since the pressure does not vanish on S_F, $\mathbb{P}(\nabla p)\neq 0$. We take $\mathbb{P}$ to the momentum equation. Then,

$$\mathbb{P}(D_tv)+\mathscr{A}v+\nabla\mathscr{H}(\eta-F(\eta))=0,\quad \mathscr{A}v=-\mathbb{P}\Delta v+\nabla\mathscr{H}(\hat{n}\cdot T_v\cdot\hat{n}). \quad (2.8)$$

The second order differential operator $\mathscr{A}$ satisfies a nice integration property: for divergence-free vector fields v, w,

$$\int_{\Omega_t}(\mathscr{A}v\cdot w)dV=\int_{\Omega_t}w\cdot(-\mathbb{P}\Delta v+\nabla\mathscr{H}(\hat{n}\cdot T_v\cdot\hat{n}))=<v,w>. \quad (2.9)$$

We need this nonnegative property of the operator $\mathscr{A}$ to obtain higher energy estimates in this section. By taking the divergence to the original equation, the Lagrangian multiplier $p_{v,v}$ can be expressed in terms of $\mathbb{P}$ as $\nabla p_{v,v}=(I-\mathbb{P})\nabla p$, and it satisfies the following elliptic system:

$$\begin{cases}-\Delta p_{v,v}=\partial_jv_i\partial_iv_j \quad \text{in} \quad \Omega_t,\\ p=0 \quad \text{on} \quad S_F, \quad \nabla p\cdot\hat{n}=-(\Delta v)\cdot\hat{n} \quad \text{on} \quad S_B,\end{cases}$$

where the last boundary condition is obtained by taking the inner product to the equation with the normal vector at the bottom.

2.3. **Regularity of the boundary.** We study the pressure to obtain the regularity of the boundary η. Since we will apply the second order differential operator $\mathscr{A}$ to (2.8), we assume that the velocity field v belongs to $L^\infty_t H^2 \cap L^2_t H^3$, from which $\nabla p \in L^2_t H^1_x$.

Let us assume that $\eta \in L^\infty_t H^a_x \cap L^2_t H^b_x$. We have two harmonic functions solving the following elliptic equations. First,

$$(E1) \begin{cases} -\Delta p_1 = 0 \quad \text{in} \quad \Omega_t, \\ p_1 = (v_{i,j} + v_{j,i}) n_i n_j \quad \text{on} \quad S_F, \quad \hat{n} \cdot \nabla p_1 = 0 \quad \text{on} \quad S_B. \end{cases}$$

Since $\nabla v \in L^2_t H^{\frac{3}{2}}_x$ on S_F, $\nabla \eta$ must be at least in $L^\infty_t H^{\frac{3}{2}}_x$ to guarantee that $\nabla p \in L^2_t H^1_x$. This implies that $a \geq \frac{5}{2}$. As we will see later, $\eta \in L^\infty_t H^{\frac{5}{2}}_x$ is not enough to obtain the priori estimate in Theorem 2.1. Secondly,

$$(E2) \begin{cases} -\Delta p_2 = 0 \quad \text{in} \quad \Omega_t, \\ p_2 = \eta - F(\eta) \quad \text{on} \quad S_F, \quad \hat{n} \cdot \nabla p_2 = 0 \quad \text{on} \quad S_B. \end{cases}$$

Since $\nabla p_2 \in L^2_t H^1_x$, $t = \frac{7}{2}$. By these two elliptic equations, we conclude that $\eta \in L^\infty_t H^{\frac{5}{2}+}_x \cap L^2_t H^{\frac{7}{2}}_x$. From the evolution equation of η, we deduce that $\eta_t \in L^2_t H^{\frac{5}{2}}_x$, and combined with $\eta \in L^2_t H^{\frac{7}{2}}_x$, this implies that $\eta \in L^\infty_t H^3_x$. These higher regularity of η are obtained by surface tension.

Now, we rewrite (2.8) as a sum of linear and nonlinear terms.

$$\begin{aligned} v_t + \mathscr{A} v + \nabla \mathscr{H}(\eta - \Delta_0 \eta) &= -\mathbb{P}(v \cdot \nabla v) + \nabla \mathscr{H}(-\Delta_0 \eta + F(\eta)) \\ &- \mathbb{P} \nabla \cdot (v \otimes v) + \nabla \mathscr{H}\Big(\frac{\nabla \eta |\nabla \eta|^2}{\sqrt{1+|\nabla \eta|^2}(1+\sqrt{1+|\nabla \eta|^2})}\Big). \end{aligned} \tag{2.10}$$

The right-hand side of (2.10) is derivatives of quadratic nonlinear terms. By the regularity $v \in L^\infty_t H^2 \cap L^2_t H^3$ and $\eta \in L^\infty_t H^3_x \cap L^2_t H^{\frac{7}{2}}_x$, the right-hand side of (2.10) is in $\nabla(L^2_t H^2_x)$. Conversely, if the right-hand side of (2.10) is in $\nabla(L^2_t H^2_x)$, then we can take two derivatives to (2.10). By acting the second order differential operator $\mathscr{A}$ to (2.10), we can establish exactly the same regularity mentioned before, and we can make the argument close.

2.4. **Higher energy estimate.** We go back to (2.8). We cannot take the usual partial derivatives to the system because it is not translation-invariant under the influence of the moving boundary. To obtain higher energy estimates, we need to use the structure of the equation. Here's one example: suppose that the heat equation is posed on a fixed domain. We can take ∂_t to the equation because the equation is translation-invariant in-time. From the equation, we see that Δ has the same effect of ∂_t, and therefore, we ca obtain higher energy bounds by applying Δ to the equation. In (2.8), the material derivative, $D_t = \partial_t + v \cdot \nabla$, corresponds to $\mathscr{A}$ so that we can apply the second order differential operator $\mathscr{A}$ to (2.8) to obtain higher energy estimates. Since $\mathscr{A}$ does not commute with the projection $\mathbb{P}$,

$$\mathscr{A}(D_t v) + \mathscr{A}(\mathscr{A} v) + \mathscr{A}(\nabla \mathscr{H}(\eta - F(\eta))) = -\mathscr{A}\Big(v \cdot \nabla v - \mathbb{P}(v \cdot \nabla v)\Big).$$

By commuting D_t with $\mathscr{A}$,

$$D_t(\mathscr{A} v) + \mathscr{A}(\mathscr{A} v) + \mathscr{A}(\nabla \mathscr{H}(\eta - F(\eta))) = [D_t, \mathscr{A}]v - \mathscr{A}(I - \mathbb{P}) v \cdot \nabla v. \tag{2.11}$$

where $\mathscr{A}(\nabla\mathscr{H}(\eta - F(\eta))) = \nabla\mathscr{H}\Big(\hat{n}\cdot T_{\nabla\mathscr{H}(\eta-F(\eta))}\cdot\hat{n}\Big)$. We multiply (2.11) by $\mathscr{A}v$ and integrate over Ω_t. Then,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\mathscr{A}v\|_{L^2}^2 + \frac{1}{2} < \mathscr{A}v, \mathscr{A}v > - \int_{\Omega_t}\mathscr{A}v\cdot\mathscr{A}\Big(v\cdot\nabla v - \mathbb{P}(v\cdot\nabla v)\Big)dV \\ &= \int_{\Omega_t}[D_t,\mathscr{A}]v\cdot\mathscr{A}v dV + \int_{\Omega_t}\mathscr{A}v\cdot\nabla\mathscr{H}\Big(\hat{n}\cdot T_{\nabla\mathscr{H}(\eta-F(\eta))}\cdot\hat{n}\Big)dV.\end{aligned} \tag{2.12}$$

Integrating (2.12) in-time,

$$\begin{aligned}&\|\mathscr{A}v(t)\|_{L^2}^2 + \int_0^t < \mathscr{A}v, \mathscr{A}v > ds - \int_0^t\int_{\Omega_s}\mathscr{A}v\cdot\mathscr{A}\Big(v\cdot\nabla v - \mathbb{P}(v\cdot\nabla v)\Big)dVds \\ &\lesssim \epsilon + \int_0^t\int_{\Omega_s}[D_t,\mathscr{A}]v\cdot\mathscr{A}v dVds + \int_0^t\int_{\Omega_s}\mathscr{A}v\cdot\nabla\mathscr{H}\Big(\hat{n}\cdot T_{\nabla\mathscr{H}(\eta-F(\eta))}\cdot\hat{n}\Big)dVds.\end{aligned} \tag{2.13}$$

First of all, we estimate the following term:

$$\int_0^t\int_{\Omega_s}\mathscr{A}v\cdot\nabla\mathscr{H}\Big(\hat{n}\cdot T_{\nabla\mathscr{H}(\eta-F(\eta))}\cdot\hat{n}\Big)dVds.$$

Since we need to estimate η in $L_t^\infty H_x^3$, we single out non-negative terms with higher order error terms.

$$\begin{aligned}&\int_{\Omega_t}\mathscr{A}v\cdot\nabla\mathscr{H}\Big(\hat{n}\cdot T_{\nabla\mathscr{H}(\eta-F(\eta))}\cdot\hat{n}\Big) = \int_{S_F}(\hat{n}\cdot\mathscr{A}v)\Big(\hat{n}\cdot T_{\nabla\mathscr{H}(\eta-F(\eta))}\cdot\hat{n}\Big) \\ &= \int_{\partial\Omega_t}\frac{\Delta_0\eta_t}{\sqrt{1+|\nabla\eta|^2}}\Delta_0(\eta-\Delta_0\eta) + (\alpha) \\ &= \frac{1}{2}\frac{d}{dt}\int_{R^2}\Big(|\Delta_0\eta|^2 + |\nabla\Delta_0\eta|^2\Big)dxdy + (\alpha).\end{aligned} \tag{2.14}$$

By (2.14), we can rewrite (2.13) as

$$\begin{aligned}&\|\mathscr{A}v(t)\|_{L^2}^2 + \int_0^t < \mathscr{A}v, \mathscr{A}v > ds + \|\Delta_0\eta(t)\|_{L^2}^2 + \|\nabla\Delta_0\eta(t)\|_{L^2}^2 \\ &\lesssim \epsilon + \int_0^t\int_{\Omega_s}[D_t,\mathscr{A}]v\cdot\mathscr{A}v dVds - \int_0^t(\alpha)ds \\ &+ \int_0^t\int_{\Omega_s}\mathscr{A}v\cdot\mathscr{A}\Big(v\cdot\nabla v - \mathbb{P}(v\cdot\nabla v)\Big)dVds.\end{aligned} \tag{2.15}$$

Next, we estimate the last integral in (2.15). By Lemma 6.3 and Corollary 6.4,

$$\begin{aligned}&\int_{\Omega_t}\mathscr{A}v\cdot\mathscr{A}(I-\mathbb{P})(v\cdot\nabla v)dV \lesssim \|\mathscr{A}v\|_{L^2}\|\mathscr{A}\Big(v\cdot\nabla v - \mathbb{P}(v\cdot\nabla v)\Big)\|_{L^2} \\ &\lesssim \|\mathscr{A}v\|_{L^2}^2 + \frac{1}{2}\|\partial^2(v\cdot\nabla v)\|_{L^2}^2 \lesssim |||v|||^4 + \frac{1}{2}\|\nabla\mathscr{A}v\|_{L^2}^2,\end{aligned} \tag{2.16}$$

where we define a norm $|||\cdot|||$ as

$$|||v||| = \|v\|_{L_t^\infty L_x^2} + \|\mathscr{A}v\|_{L_t^\infty L_x^2} + \|\nabla v\|_{L_t^2 L_x^2} + \|\nabla\mathscr{A}v\|_{L_t^2 L_x^2}.$$

Combining (2.15) and (2.16), with the basic energy estimate (2.7), we have

$$\begin{aligned}&\|v\|^2_{L^\infty_t L^2_x} + \|\nabla v\|^2_{L^2_t L^2_x} + \|\mathscr{A} v\|^2_{L^\infty_t L^2_x} + \int < \mathscr{A} v, \mathscr{A} v > dt + \|\eta\|^2_{L^\infty_t H^3_x} \\ &\lesssim \epsilon + \Big| \int \int_{\Omega_s} \mathscr{A} v \cdot ([D_t, \mathscr{A}]v) dV dt \Big| + \Big| \int (\alpha) dt \Big| + |||v|||^4 + \frac{1}{2} \|\nabla \mathscr{A} v\|^2_{L^2}.\end{aligned} \tag{2.17}$$

We have two more terms to be estimated: the commutator term $[D_t, \mathscr{A}]$ and (α). In this section we only present sketches of proofs and for details, see section 6.

Proposition 2.3. *Commutator Estimate.*

$$\int \int_{\Omega_s} \mathscr{A} v \cdot ([D_t, \mathscr{A}]v) dV dt \lesssim |||v|||^3 + \frac{1}{2} \|\nabla(\mathscr{A} v)\|^2_{L^2_t L^2_x}. \tag{2.18}$$

Proposition 2.4. *Estimation of* (α).

$$\int (\alpha) dt \lesssim \|\eta\|^2_{L^\infty_t H^3_x} \Big(|||v|||^2 + \|\nabla \mathscr{H}(\eta - F(\eta))\|^2_{L^2_t H^1_x} \Big) + |||v|||^4 \Big(1 + \|v_t\|^2_{L^2_t H^1_x} \Big) \tag{2.19}$$

To prove Proposition 2.4, we need the following estimate. For details, see section 6.

Proposition 2.5. *Estimation of* $\|v_t\|^2_{L^2_t H^1_x} + \|\nabla \mathscr{H}(\eta - F(\eta))\|^2_{L^2_t H^1_x}$.

$$\begin{aligned}&\|v_t\|^2_{L^2_t H^1_x} + \|\nabla \mathscr{H}(\eta - F(\eta))\|^2_{L^2_t H^1_x} \\ &\lesssim \epsilon + \|\eta\|^2_{L^\infty_t H^3_x} \Big(|||v|||^2 + \|\nabla \mathscr{H}(\eta - F(\eta))\|^2_{L^2_t H^1_x} \Big) + |||v|||^4 \|v_t\|^2_{L^2_t H^1_x}.\end{aligned} \tag{2.20}$$

In the right-hand side of (2.17), we have the full derivative of $\mathscr{A} v$, while we only have the symmetric part of $\mathscr{A} v$ in the left-hand side of (2.17). Therefore, we need a Korn-type inequality to move $\frac{1}{2}\|\nabla \mathscr{A} v\|^2_{L^2}$ in the right-hand side of (2.17) to the left-hand side of (2.17).

Proposition 2.6. *Korn-type Inequality.*

$$\begin{aligned}\|\nabla \mathscr{A} v\|^2_{L^2_t L^2_x} \lesssim &\int < \mathscr{A} v, \mathscr{A} v > dt + |||v|||^4 + \frac{1}{2} \|\nabla \mathscr{A} v\|^2_{L^2_t L^2_x} \\ &+ \|\nabla \mathscr{H}(\eta - F(\eta))\|^2_{L^2_t H^1_x}.\end{aligned} \tag{2.21}$$

Now, we can derive the energy bound in Theorem 2.1. By Proposition 2.6, we can replace $\|\nabla \mathscr{A} v\|^2_{L^2_t L^2_x}$ by $\int < \mathscr{A} v, \mathscr{A} v > dt$ in (2.17). By Proposition 2.5,

$$\begin{aligned}&\|v_t\|^2_{L^2_t H^1_x} + \|\nabla \mathscr{H}(\eta - F(\eta))\|^2_{L^2_t H^1_x} + |||v|||^2 + \|\eta\|^2_{L^\infty_t H^3_x} \\ &\lesssim \epsilon + \Big(|||v|||^2 + \|\eta\|^2_{L^\infty_t H^3_x} + \|v_t\|^2_{L^2_t H^1_x} + \|\nabla \mathscr{H}(\eta - F(\eta))\|^2_{L^2_t H^1_x} \Big)^2.\end{aligned} \tag{2.22}$$

where $\epsilon = \|v_0\|^2_{H^2} + \|\eta_0\|^2_{H^3}$. By Lemma 6.3 and Corollary 6.4, (2.17) implies

$$\begin{aligned}&\|v\|^2_{L^\infty_t H^2_x} + \|\nabla v\|^2_{L^2_t H^2_x} + \|\eta\|^2_{L^\infty_t H^3_x} + \|\nabla \mathscr{H}(\eta - F(\eta))\|^2_{L^2_t H^1_x} \\ &\lesssim \epsilon + \Big(\|v\|^2_{L^\infty_t H^2_x} + \|\nabla v\|^2_{L^2_t H^2_x} + \|\eta\|^2_{L^\infty_t H^3_x} + \|\nabla \mathscr{H}(\eta - F(\eta))\|^2_{L^2_t H^1_x} \Big)^2.\end{aligned} \tag{2.23}$$

This completes the proof of Theorem 2.1.

3. **Existence and uniqueness on the equilibrium domain.** In this section we prove the main result, Theorem 1.3, as we mentioned in the introduction, by using Beale's method [4] of solving the problem on the equilibrium domain. Since we will project the equation onto the divergence-free space to obtain L^2 in-time bounds of the boundary, we need to keep the divergence-free condition to the velocity field on the equilibrium domain. Therefore, the transformation from the moving domain into the equilibrium domain is given by the change of variables in a way that the divergence-free condition is preserved. Let us briefly explain the idea of proof. After deriving the system of equations on the equilibrium domain, we will obtain the a priori estimate under the assumption that we can solve the linearized problem. Once we obtain the a priori estimate, we can iterate the system to finish the proof. The solvability of the linearized problem will be the subject of the next section.

To this end, we first define a map

$$\theta(t) : \Omega = \{(x, y, z); -1 < z < 0\} \to \{(x, y, z^{'}); -1 < z^{'} < \eta(x, y, t)\},$$

by using the harmonic extension $\bar{\eta}$ of η in the following way:

$$\theta(x, y, z, t) = (x, y, \bar{\eta}(x, y, t) + z(1 - \bar{\eta}(x, y, t))). \tag{3.1}$$

In order θ to be a diffeomorphism, η should be small for all time. This smallness condition will be achieved by higher energy estimates. We define v on $\theta(\Omega)$ by

$$v_i = \frac{\theta_{i,j}}{J} w_j = \alpha_{ij} w_j, \quad J = 1 - \bar{\eta} + \partial_z \bar{\eta}(1 - z), \quad d\theta = (\theta_{i,j}).$$

Then, v is divergence-free in $\theta(\Omega)$ if and only if w has the same property in Ω. We replace the system of equations of v with that of w.

$$v_{i,j} = \zeta_{lj} \partial_l (\alpha_{ik} w_k), \quad v_{i,t} = \alpha_{ij} w_{j,t} + \alpha_{ij}^{'} w_j + (\theta^{-1})_3^{'} \partial_3 (\alpha_{ij} w_j),$$

where $\zeta = (d\theta)^{-1}$ and $'$ denotes derivatives in t. Setting $q = p \circ \theta$, the other three terms in the Navier-Stokes equations are of the form

$$\alpha_{jk} w_k \zeta_{mj} \partial_m (\alpha_{il} w_l) - \zeta_{kj} \partial_k (\zeta_{mj} \partial_m (\alpha_{il} w_l)) + \zeta_{ki} \partial_k q. \tag{3.2}$$

Multiplying (3.2) by $(\alpha_{ij})^{-1}$, we have the following system of equations:

$$w_t - \Delta w + \nabla q = f(\bar{\eta}, v, \nabla q). \tag{3.3}$$

The normal boundary condition becomes

$$qN_i - \Big(\zeta_{lj} \partial_l (\alpha_{ik} w_k) + \zeta_{mi} \partial_m (\alpha_{jk} w_k)\Big) N_j = \Big(\eta - \nabla \cdot (\frac{\nabla \eta}{\sqrt{1 + |\nabla \eta|^2}})\Big) N_i, \tag{3.4}$$

where $\hat{N} = \hat{n} \circ \theta$. Let $\hat{T}_1 = (1, 0, \partial_x \eta)$, $\hat{T}_2 = (0, 1, \partial_y \eta)$. Taking the inner product to (3.4) with $\hat{T}_1$, $\hat{T}_2$, and $\hat{N}$, we obtain that

$$w_{i,3} + w_{3,i} = g_i(\eta, w), \quad q - w_{3,3} = \eta - \Delta_0 \eta + g_3, \tag{3.5}$$

where $g_3 = \Delta\eta - F(\eta) + g_3^{'}$, $g_3^{'}$ is quadratic in η and w. Finally, the evolution equation of η can be obtained in terms of the new velocity field on Ω:

$$\eta_t = w_3 \quad \text{on} \quad \{z = 0\}. \tag{3.6}$$

In sum, we have the following linearized system of equations on the equilibrium domain:

$$(LNSF)\begin{cases} w_t - \Delta w + \nabla q = f \quad \text{in} \quad \Omega, \\ \nabla\cdot w = 0 \quad \text{in} \quad \Omega, \\ w_{i,3} + w_{3,i} = g_i \quad \text{on} \quad \{z=0\}, \quad \eta_t = w_3 \quad \text{on} \quad \{z=0\}, \\ q = w_{3,3} + \eta - \Delta_0\eta + g_3 \quad \text{on} \quad \{z=0\}, \\ w = 0 \quad \text{on} \quad \{z=-1\}, \end{cases}$$

and the corresponding compatibility conditions of the initial data on the equilibrium domain are given by

$$\begin{cases} \nabla\cdot w_0 = 0 \quad \text{in} \quad \Omega, \\ w_0 = 0 \quad \text{on} \quad \{z=-1\}, \\ w(0)_{i,3} + w(0)_{3,i} = g_i(0), \quad q(0) = w(0)_{3,3} + g_3(0) \quad \text{on} \quad \{z=0\}. \end{cases}$$

Suppose that (LNSF) is solvable. (We will prove the solvability of the linearized problem in section 4.) Let

$$|||(w,\eta,q)||| = \|w\|_{L^\infty_t H^2_x} + \|w\|_{L^2_t H^3_x} + \|w_t\|_{L^2_t H^1_x} + \|\eta\|_{L^\infty_t H^3_x} + \|\nabla q\|_{L^2_t H^1_x}.$$

Then, by Proposition 4.1,

$$|||(w,\eta,q)||| \lesssim \epsilon + \|f\|_{L^2_t H^1_x} + \|f_t\|_{L^2_t L^2_x} + \|g\|_{L^2_t H^{\frac{3}{2}}_x} + \|g_t\|_{L^2_t \dot{H}^{-\frac{1}{2}}_x} + \|g\|^2_{L^\infty_t H^{\frac{1}{2}}_x} \tag{3.7}$$

Now, we calculate nonlinear terms. Principal parts are given by

$$f \sim w\nabla^3\bar{\eta} + \nabla^2\bar{\eta}\nabla w + \nabla^2 w\nabla\bar{\eta} + \nabla\bar{\eta}\nabla q, \quad g_i \sim \nabla\eta\nabla w + \nabla(\nabla\eta\nabla\eta). \tag{3.8}$$

We only need to estimate the highest order terms.

$$\begin{aligned} &\|\nabla f\|_{L^2_t L^2_x} \lesssim \|\partial(w\nabla^3\bar{\eta}) + \partial(\nabla^2\bar{\eta}\nabla w) + \partial(\nabla^2 w\nabla\bar{\eta}) + \partial(\nabla q\nabla\bar{\eta})\|_{L^2_t L^2_x} \\ \lesssim& \|\nabla w\nabla^3\bar{\eta}\|_{L^2_t L^2_x} + \|w\nabla^4\bar{\eta}\|_{L^2_t L^2_x} + \|\nabla w\nabla^3\bar{\eta}\|_{L^2_t L^2_x} + \|\nabla^2 w\nabla^2\bar{\eta}\|_{L^2_t L^2_x} \\ &+ \|\nabla^3 w\nabla\bar{\eta}\|_{L^2_t L^2_x} + \|\nabla^2 w\nabla^2\bar{\eta}\|_{L^2_t L^2_x} + \|\nabla^2 q\nabla\bar{\eta}\|_{L^2_t L^2_x} + \|\nabla q\nabla^2\bar{\eta}\|_{L^2_t L^2_x} \\ \lesssim& \|\nabla w\|_{L^2_t L^\infty_x}\|\nabla^3\bar{\eta}\|_{L^\infty_t L^2_x} + \|w\|_{L^\infty_t L^\infty_x}\|\nabla^4\bar{\eta}\|_{L^2_t L^2_x} \\ &+ \|w\|_{L^2_t H^3_x}\|\nabla\bar{\eta}\|_{L^\infty_t L^\infty_x} + \|w\|_{L^\infty_t H^2_x}\|\nabla^2\bar{\eta}\|_{L^2_t L^\infty_x} \\ &+ \|\nabla^2 q\|_{L^2_t L^2_x}\|\nabla\bar{\eta}\|_{L^\infty_t L^\infty_x} + \|\nabla q\|_{L^2_t L^2_x}\|\nabla^2\bar{\eta}\|_{L^\infty_t L^\infty_x}. \end{aligned} \tag{3.9}$$

By Lemma 6.7, we can replace $\|\nabla^4\bar{\eta}\|_{L^2_t L^2_x}$ and $\|\nabla^2\bar{\eta}\|_{L^2_t L^\infty_x}$ by $\|\nabla\mathscr{H}(\eta-\Delta_0\eta)\|_{L^2_t H^1_x}$. We can do the same calculation to $\|f_t\|_{L^2_t L^2_x}$. Hence, $\|f\|_{L^2_t H^1_x} + \|f_t\|_{L^2_t L^2_x} \lesssim |||(w,\eta,q)|||^2$. We do the same calculation to g.

$$\begin{aligned} &\|\partial^{\frac{3}{2}} g\|_{L^2_t L^2_x(R^2)} \lesssim \|\partial^{\frac{3}{2}}(\nabla w)\nabla\eta + \nabla w\partial^{\frac{3}{2}}(\nabla\eta) + \nabla^{\frac{5}{2}}(\nabla\eta\nabla\eta)\|_{L^2_t L^2_x(R^2)} \\ \lesssim& \|\partial^{\frac{3}{2}}(\nabla w)\|_{L^2_t L^2_x(R^2)}\|\nabla\eta\|_{L^\infty_t L^\infty_x} + \|\nabla w\|_{L^2_t L^\infty_x(R^2)}\|\partial^{\frac{5}{2}}\eta\|_{L^\infty_t L^2_x} \\ &+ \|\nabla\eta\|_{L^\infty_t L^\infty_x}\|\partial^{\frac{7}{2}}\eta\|_{L^2_t L^2_x}. \end{aligned} \tag{3.10}$$

Therefore, $\|g\|_{L^2_t H^{\frac{3}{2}}_x(R^2)} \lesssim |||(w,\eta,q)|||^2$. Again, we can do the same calculation to $\|g_t\|_{L^2_t \dot{H}^{-\frac{1}{2}}_x}$ and $\|g\|_{L^\infty_t H^{\frac{1}{2}}_x}$: $\|g_t\|_{L^2_t \dot{H}^{-\frac{1}{2}}_x} + \|g\|_{L^\infty_t H^{\frac{1}{2}}_x} \lesssim |||(w,\eta,q)|||^2$. From (3.7), we conclude that

$$|||(w,\eta,q)||| \lesssim \epsilon + (|||(w,\eta,q)|||)^2.$$

Once we obtain the above energy estimate on the fixed domain, we can iterate the system. The first step is to define the first iteration (w^1,η^1,q^1) in terms of the initial data. Then, we can define the second step (w^2,η^2,q^2), so on. Let $\rho(t)$ be a nice cut-off function in-time such that $\rho(0) = 1$. $\mu(x, y)$ is a $\mathscr{C}_c^\infty$ function on R^2. Then η^1 is defined as $\eta^1 = (\eta_0 \star \mu(x, y))\rho(t)$. We do the same procedure to define the first velocity field w^1. Since w_0 is defined on the channel Ω, we restrict w_0 interior of the domain. We choose a $\mathscr{C}^\infty$ function ψ supported in $-\frac{3}{4} < z < -\frac{1}{4}$. Let $\phi(x, y, z)$ be a nice function such that $\phi = 0$ outside of the domain. Let $\lambda(x, y, z)$ be a $\mathscr{C}_c^\infty$ function on R^3. Then, we define w^1 as $w^1 = ((w_0\psi) \star \lambda(x, y, z))\phi(x, y, z)\rho(t)$. Finally, we define the pressure as $q^1 = \mathscr{H}(w^1_{3,3} + \eta^1 - \Delta\eta^1)$. Then, for $n > 1$, we iterate the system of equations in the following manner:

$$(LNSF^m) \begin{cases} w^m_t - \Delta w^m + \nabla q^m = f(w^{m-1}, \eta^{m-1}, q^{m-1}) \quad \text{in} \quad \Omega, \\ \nabla \cdot w^m = 0 \quad \text{in} \quad \Omega, \\ w^m_{i,3} + w^m_{3,i} = g_i(w^{m-1}, \eta^{m-1}) \quad \text{on} \quad \{z = 0\}, \\ q^m = w^m_{3,3} + \eta^m - \Delta_0\eta^m + g_3(w^{m-1}, \eta^{m-1}) \quad \text{on} \quad \{z = 0\}, \\ \eta^m_t = w^m_3 \quad \text{on} \quad \{z = 0\}, \quad w^m = 0 \quad \text{on} \quad \{z = -1\}. \end{cases}$$

From the a priori estimate in Proposition 4.1,

$$|||(w^m, \eta^m, q^m)||| \lesssim \epsilon + |||(w^{m-1}, \eta^{m-1}, q^{m-1})|||^2.$$

Therefore, we conclude that $\{|||(w^m, \eta^m, q^m)|||\}$ are uniformly bounded if the initial data is small enough. By taking difference of two sequences, we can show that $\{(w^m, \eta^m, q^m)\}$ is a Cauchy sequence with respect to the norm defined in Theorem 1.3. Therefore, we can obtain a unique, global-in-time solution to (LNSF) if initial data is small enough in H^2 by the contraction mapping theorem. This completes the proof of Theorem 1.3. The dependence on the initial data of the boundary η_0 occurs when we define the first iteration.

4. **Solvability of (LNSF).** In this section we study the linearized problem (LNSF) defined on the equilibrium domain $\Omega = \{(x, y, z); -1 < z < 0\}$. When we define the new system of equations on Ω in section 3, we made the new velocity field of divergence-free. We need to keep the divergence-free condition to the velocity field from two reasons. First of all, we need to remove the interior pressure in the weak formulation because we only know the explicit form of the pressure at the boundary. By the integration by parts,

$$(\nabla p, \phi) = \int_{\partial\Omega} p(\phi \cdot \hat{n})dS - (p, \nabla \cdot \phi).$$

Therefore, to remove $(p, \nabla \cdot \phi)$ we have to define test functions in the divergence-free space. Secondly, as we remark at the end of this section, we can estimate the boundary in L^2 in-time by projecting the momentum equation onto the divergence-free space. (See Remark 3).

In this section we first prove that (LNSF) is solvable weakly in L^2 for given initial data and for given external forces f and g, and then improve the regularity of weak solutions under higher regularity of the initial data and (f, g) in Proposition 4.1. In this section the upper index means the third component of a vector field.

4.1. **Weak formulation.** First, we define a function space where weak solutions will be defined. For any fixed time interval $[0, T]$ with $T < \infty$,

$$\mathscr{V}(T) = \Big\{ v \in L^2_t H^1_x : \ \nabla \cdot v = 0, \ \int_0^t v^3 ds \in L^\infty_t L^2_x(R^2), \ \int_0^t \nabla_0 v^3 ds \in L^\infty_t L^2_x(R^2) \Big\},$$

with $v = 0$ on S_B. The divergence-free condition is expressed in the distributional form, i.e. v is orthogonal to gradients of test functions which vanish on S_F. This space is almost the same space used for the Navier-Stokes equations in a fixed domain except for the boundary terms: $\int_0^t v^3 ds$, $\int_0^t \nabla_0 v^3 ds$.

For test functions, we define a separable space $\mathscr{V}$ as

$$\mathscr{V} = \Big\{ v \in H^1_x : \nabla \cdot v = 0, \quad v = 0 \quad \text{on} \quad S_B, \quad v^3 \in L^2(R^2), \quad \nabla_0 v^3 \in L^2(R^2) \Big\}.$$

We define a space for v_t:

$$\mathscr{V}'(T) = \Big\{ v \in L^2_t H^{-1}_x : \nabla \cdot v = 0, \quad v = 0 \quad \text{on} \quad S_B \Big\}.$$

We say that $(w, w_t) \in \mathscr{V}(T) \times L^2(0, T; \mathscr{V}')$ is a weak solution of (LNSF) if for all $v \in \mathscr{V}$, $\nabla \cdot w = 0 \quad w(\cdot, 0) = w_0 \in L^2$ hold and

$$\begin{aligned} &(w_t, v) + < w, v > + \int_{R^2} \Big((\int_0^t w^3 ds) v^3 \Big) + \int_{R^2} \Big(\nabla_0 (\int_0^t w^3 ds) \cdot (\nabla_0 v^3) \Big) \\ &= (f, v) + (g, v). \end{aligned} \tag{4.1}$$

4.2. **Existence of weak solutions.** Here, we want to address the following existence theorem:

For any $w_0 \in L^2$, $f \in L^2_t L^2_x(\Omega)$, $g \in L^2_t L^2_x(R^2)$, there exists a weak solution $(w, w_t) \in \mathscr{V}(T) \times \mathscr{V}'(T)$ such that $w(\cdot, 0) = w_0$.

The idea of obtaining a weak solution is quite standard. Since $\mathscr{V}$ is separable, we can use the Galerkin approximation to the equation, from which we can solve an ODE to decide the coefficients in a fixed time interval $[0, T]$. Then, we obtain the uniform energy estimate to these approximated equations, from which we can pass to the limit. By taking a cut-off function in-time, we can prove that a weak solution achieves the initial data in L^2.

▶ **Galerkin Approximation:** Since $\mathscr{V}$ is separable, there exists an orthogonal basis $\big\{\phi_k\big\}$ in L^2. By approximating w by $w_m(t) = \sum_{j=1}^m \lambda^j_m(t) \phi_j$, we want to select the coefficients $\lambda^j_m(t)$ such that $\lambda^j_m(0) = (w_0, \phi_j)$ and

$$\begin{aligned} &(\partial_t w_m, \phi_j) + < w_m, \phi_j > + \int \int_0^t w^3_m \phi^3_j dx ds + \int \nabla_0 \int_0^t w^3_m \cdot \nabla_0 \phi^3_j dx ds \\ &= (f, \phi_j) + (g, \phi_j). \end{aligned} \tag{4.2}$$

We define integrals as

$$E_{mj} = < \phi_m, \phi_j >, \quad H_{mj} = \int_{R^2} (\phi^3_m)(\phi^3_j) dx dy, \quad L_{mj} = \int_{R^2} \nabla_0 (\phi^3_m) \cdot (\nabla_0 \phi^3_j) dx dy,$$

$$F_j = (f, \phi_j), \quad G_j = (g, \phi_j).$$

Since $(\partial_t w_m, \phi_j) = \partial_t \lambda^j_m$, (4.2) is reduced to an ODE,

$$\partial_t \lambda^j_m + E_{mj} \lambda^j_m + H_{mj} \int_0^t \lambda^j_m(s) ds + L_{mj} \int_0^t \lambda^j_m(s) ds = F_j + G_j,$$

which is subject to the initial data $\lambda^j_m(0) = (w_0, \phi_j)$. By the standard existence theory for ODE, there exists a unique absolutely continuous function $\lambda_m(t) = \big\{ \lambda^j_m : j = 1, 2, \cdots, m \big\}$.

▶ **Energy estimate:** For each $m = 1, 2, \cdots$,

$$\begin{aligned}&\|w_m\|^2_{L^\infty_t L^2_x} + \|\nabla w_m\|^2_{L^2_t L^2_x} + \|\int_0^t w^3_m ds\|^2_{L^\infty_t L^2_x} + \|\nabla_0 \int_0^t w^3_m ds\|^2_{L^\infty_t L^2_x} \\ &\lesssim \|w_0\|^2_{L^2_x} + \|f\|^2_{L^2_t L^2_x} + \|g\|^2_{L^2_t L^2_x}.\end{aligned} \tag{4.3}$$

Proof. We multiply (4.2) by $\lambda^j_m(t)$ and sum for $j = 1, 2, \cdots, m$.

$$\begin{aligned}&(w_m^{'}, w_m) + < w_m, w_m > + \int (\int_0^t w^3_m)(w^3_m) dxds \\ &+ \int (\nabla_0 \int_0^t w^3_m) \cdot \nabla_0 (w^3_m) dxds \\ &= \frac{1}{2}\frac{d}{dt}\|w_m\|^2_{L^2} + < w_m, w_m > + \frac{1}{2}\frac{d}{dt}\Big(\int_{R^2} |(\int_0^t w^3_m ds)|^2 dx\Big) \\ &+ \frac{1}{2}\frac{d}{dt}\Big(\int_{R^2} |(\nabla_0 \int_0^t w^3_m ds)|^2 dxdy\Big) \\ &= (f, w_m) + (g, w_m) \lesssim \|f\|^2_{L^2} + \|g\|^2_{L^2} + \frac{1}{2}\|w_m\|^2_{L^2}.\end{aligned} \tag{4.4}$$

Since $w = 0$ at the bottom, we can replace $\|w\|^2_{L^2_t L^2_x}$ in the last line of (4.4) by $\|\nabla w\|^2_{L^2_t L^2_x}$. Integrating (4.4) in-time, we obtain (4.3). □

▶ **Passing to the limit:** From the energy estimate, we know that $\{w_m\}$ is uniformly bounded in $L^\infty_t L^2_x \cap L^2_t H^1_x$. Therefore, there exists a subsequence, still denoted by $\{w_m\}$, converging to w for the weak star topology in $L^\infty_t L^2_x$ and for the weak topology in $L^2_t H^1_x$. Since $\{w_m\}$ is bounded in $\mathscr{V}(T)$, for the weak star topology in $L^\infty_t H^1_x$, $(\int_0^t w^3_m ds)$ and $(\nabla_0 \int_0^t w^3_m ds)$ converge to $\int_0^t w^3 ds$ and $\nabla_0 \int_0^t w^3 ds$, respectively. We multiply (4.2) $\psi \in \mathscr{D}(0, T)$ such that $\psi(T) = 0$, and integrate in-time. By the integration by parts in-time,

$$\begin{aligned}&- \int_0^T (w_m, \phi_j)\partial_t \psi dt + \int_0^T < w_m, \psi(t)\phi_j > dt + \int_0^T \int_{R^2} (\int_0^t w^3_m ds)(\psi(t)\phi^3_j) dxdt \\ &+ \int_0^T \int_{R^2} \nabla_0 (\int_0^t w^3_m ds) \cdot (\psi(t)\nabla_0 \phi^3_j) dxdt \\ &= (w_m(0), \phi_j)\psi(0) + \int_0^T \Big((f, \phi_j) + (g, \phi_j)\Big) dt.\end{aligned} \tag{4.5}$$

Since $w_m(0) \to w_0$ in L^2, by letting $m \to \infty$ in (4.5),

$$\begin{aligned}&- \int_0^T (w, \phi_j)\partial_t \psi dt + \int_0^T < w, \psi(t)\phi_j > + \int_0^T \int (\int_0^t w^3)(\psi(t)\phi^3_j) dxdt \\ &+ \int_0^T \int \nabla_0 (\int_0^t w^3) \cdot (\psi(t)\nabla_0 \phi^3_j) dxdt \\ &= (w_0, \phi_j)\psi(0) + \int_0^T \Big((f, \phi_j) + (g, \phi_j)\Big) dt.\end{aligned} \tag{4.6}$$

Since (4.6) holds for a finite linear combination of $\phi_j^{,}s$, it also holds for all $v \in \mathscr{V}$. Therefore,

$$\begin{aligned}
&-\int_0^T (w,v)\partial_t\psi dt + \int_0^T < w, v\psi(t) > dt + \int_0^T \int_{R^2} (\int_0^t w^3 ds)(v^3)\psi(t) dx dt \\
&+ \int_0^T \int_{R^2} \nabla_0(\int_0^t w^3 ds)\cdot(\nabla_0 v^3)\psi(t) dx dt \qquad (4.7)\\
&= (w_0, v)\psi(0) + \int_0^T \Big((f,v) + (g,v)\Big) dt,
\end{aligned}$$

from which we can achieve the following equality,

$$\begin{aligned}
&(w_t, v) + < w, v > + \int_{R^2} (\int_0^t w^3 ds)(v^3) dx + \int_{R^2} \nabla_0(\int_0^t w^3 ds)\cdot(\nabla_0 v^3) dx \\
&= (f,v) + (g,v) \qquad (4.8)
\end{aligned}$$

in the distribution sense on $(0,T)$. It remains to show that $w(0) = w_0$ in L^2. We multiply (4.8) by $\psi(t)$, and integrate in-time.

$$\begin{aligned}
&-\int_0^T (w,v)\partial_t\psi dt + \int_0^T < w, v\psi(t) > dt + \int_0^T \int_{R^2} (\int_0^t w^3 ds)(v^3)\psi(t) dx dt \\
&+ \int_0^T \int_{R^2} \nabla_0(\int_0^t w^3 ds)\cdot(\nabla_0 v^3)\psi(t) dx dt \qquad (4.9)\\
&= (w(0), v)\psi(0) + \int_0^T \Big((f,v) + (g,v)\Big) dt.
\end{aligned}$$

Comparing (4.7) with (4.9), we see that $(w_0 - w(0), v)\psi(0) = 0$ for each $v \in \mathscr{V}$. We choose ψ such that $\psi(0) \neq 0$. Then $w(0) = w_0$. This completes the existence part.

Remark 1. Since the trace theorem does not hold in this level of the regularity of weak solutions, weak solutions are not in $L^2(0,T;\mathscr{V})$ so that we cannot take the difference of two weak solutions to show that a weak solution is unique. We can show uniqueness after proving the regularity result in Proposition 4.1.

4.3. **Higher regularity.** In this section we improve the regularity of weak solutions under higher regularity of initial data and external forces. Since the domain is translation-invariant in the horizontal direction, we need to take tangential derivatives to the equation to obtain energy bounds of tangential derivatives. Other bounds can be obtained from the divergence-free condition and from the momentum equation. As we will see in the proof, we need to control w_t to obtain estimates of the full derivatives of the velocity field, and these estimates can be established by solving an elliptic problem in the proof of Proposition 4.1.

Proposition 4.1. *Suppose that $(w, w_t) \in \mathscr{V}(T) \times \mathscr{V}'(T)$ is a weak solution such that the initial data satisfies the compatibility condition. Let $w_0 \in H^2$, $f \in L_t^2 H_x^1 \cap L_t^\infty L^2$, $g \in L_t^2 H_x^{\frac{3}{2}} \cap L_t^\infty \dot{H}^{-\frac{1}{2}}$. Then, $w \in L_t^\infty H_x^2 \cap L_t^2 H_x^3$, $\eta \in L_t^\infty H_x^3$ and $\nabla q \in L_t^2 H_x^1$. Moreover, (w, η, p) satisfies the following energy bound:*

$$\begin{aligned}
&\|w\|_{L_t^\infty H_x^2} + \|\nabla w\|_{L_t^2 H_x^2} + \|\eta\|_{L_t^\infty H_x^3} + \|\nabla q\|_{L_t^2 H_x^1} + \|w_t\|_{L_t^2 H_x^1} \\
&\lesssim \|w_0\|_{H^2} + \|\eta_0\|_{H^3} + \|f\|_{L_t^2 H_x^1} + \|f_t\|_{L_t^2 L_x^2} + \|g\|_{L_t^2 H_x^{\frac{3}{2}}} + \|g_t\|_{L_t^2 \dot{H}_x^{-\frac{1}{2}}} + \|g\|^2_{L_t^\infty H_x^{\frac{1}{2}}}.
\end{aligned}$$

Proof. It consists of 7 steps.

• **Step 1. L^2 estimate:** we multiply the momentum equation of (LNSF) by w and integrate over Ω. By the integration by parts,

$$\frac{1}{2}\frac{d}{dt}\|w\|_{L^2}^2 + < w, w > + (q - w_{3,3}, w_3) - (w_i, g_i) = (f, w). \tag{4.10}$$

From the boundary condition,

$$(q - w_{3,3}, w_3) = (\eta - \Delta_0\eta + g_3, w_3) = \frac{1}{2}\frac{d}{dt}\Big(\|\eta\|_{L^2}^2 + \|\nabla\eta\|_{L^2}^2\Big) + (g_3, w_3). \tag{4.11}$$

By the trace theorem and Young's inequality, the last term in (4.11) can be estimated as

$$(g_i, w_i) \lesssim \|g\|_{L^2}^2 + \frac{1}{2}\|w\|_{L^2(\mathbb{R}^2)}^2 \lesssim \|g\|_{L^2}^2 + \frac{1}{2}\|w\|_{L^2}^2 + \frac{1}{2}\|\nabla w\|_{L^2}^2. \tag{4.12}$$

By (4.11) and (4.12), (4.10) can be estimated by

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big(\|w\|_{L^2}^2 + \|\eta\|_{L^2}^2 + \|\nabla\eta\|_{L^2}^2\Big) + < w, w > \\ &\lesssim \|g\|_{L^2}^2 + \|f\|_{L^2}^2 + \frac{1}{2}\|w\|_{L^2}^2 + \frac{1}{2}\|\nabla w\|_{L^2}^2. \end{aligned} \tag{4.13}$$

Integrating (4.13) in-time,

$$\begin{aligned} &\|w\|_{L_t^\infty L_x^2}^2 + \int < w, w > dt + \|\eta\|_{L_t^\infty L_x^2}^2 + \|\nabla\eta\|_{L_t^\infty L_x^2}^2 \\ &\lesssim \epsilon + \|f\|_{L_t^2 L_x^2}^2 + \|g\|_{L_t^2 L_x^2}^2 + \frac{1}{2}\|w\|_{L_t^2 L_x^2}^2 + \frac{1}{2}\|\nabla w\|_{L_t^2 L_x^2}^2. \end{aligned} \tag{4.14}$$

By Korn's inequality, we can replace the symmetric part of ∇w in the left-hand side of (4.14) by the full derivative. Since $w = 0$ at the bottom, we can replace $\|w\|_{L_t^2 L_x^2}^2$ in the right-hand side of (4.14) by $\|\nabla w\|_{L_t^2 L_x^2}^2$. Therefore, we have

$$\|w\|_{L_t^\infty L_x^2}^2 + \|\nabla w\|_{L_t^2 L_x^2}^2 + \|\eta\|_{L_t^\infty L_x^2}^2 + \|\nabla\eta\|_{L_t^\infty L_x^2}^2 \lesssim \epsilon + \|f\|_{L_t^2 L_x^2}^2 + \|g\|_{L_t^2 L_x^2}^2. \tag{4.15}$$

• **Step 2. H^1 estimate:** next, we obtain bounds of derivatives by following the same argument in step 1. We multiply the momentum equation by $\Delta_0 w$ and integrate in the spatial variables. Since $\nabla_0 w = 0$ at the bottom, boundary terms on $z = 0$ only are involved when we do the integration by parts. By the integration by parts,

$$\frac{d}{dt}\|\nabla_0 w\|_{L^2}^2 + < \nabla_0 w, \nabla_0 w > + (q - w_{3,3}, \Delta_0 w_3) \lesssim \|f\|_{L^2}^2. \tag{4.16}$$

From the boundary condition,

$$\frac{d}{dt}\|\nabla_0 w\|_{L^2}^2 + < \nabla_0 w, \nabla_0 w > + (\eta - \Delta_0\eta, \Delta_0 w_3) + (\nabla_0 g_3, \nabla_0 w_3) \lesssim \|f\|_{L^2}^2. \tag{4.17}$$

By the duality argument in $(\nabla_0 g_3, \nabla_0 w_3)$ and Young's inequality,

$$\begin{aligned} &\frac{d}{dt}\|\nabla_0 w\|_{L^2}^2 + < \nabla_0 w, \nabla_0 w > + \frac{d}{dt}\Big(\|\nabla_0\eta\|_{L^2}^2 + \|\nabla_0^2\eta\|_{L^2}^2\Big) \\ &\lesssim \|\nabla_0 g_3\|_{\dot{H}^{-\frac{1}{2}}}^2 + \frac{1}{2}\|\nabla_0 w\|_{\dot{H}^{\frac{1}{2}}(R^2)}^2 + \|f\|_{L^2}^2 \\ &\lesssim \|g\|_{H^{\frac{1}{2}}}^2 + \frac{1}{2}\|\nabla_0 w\|_{H^{\frac{1}{2}}(R^2)}^2 + \|f\|_{L^2}^2. \end{aligned} \tag{4.18}$$

By the trace theorem, $\frac{1}{2}\|\nabla_0 w\|^2_{H^{\frac{1}{2}}(R^2)} \lesssim \frac{1}{2}\|\nabla_0 w\|^2_{H^1}$. Therefore,

$$\begin{aligned}&\frac{d}{dt}\Big(\|\nabla_0 w\|^2_{L^2}+\|\nabla\eta\|^2_{L^2}+\|\nabla^2\eta\|^2_{L^2}\Big)+<\nabla_0 w,\nabla_0 w>\\&\lesssim \frac{1}{2}(\|\nabla_0\nabla w\|^2_{L^2}+\|\nabla_0 w\|^2_{L^2})+\|f\|^2_{L^2}+\|g\|^2_{H^{\frac{1}{2}}}.\end{aligned} \tag{4.19}$$

By Korn's inequality, we can move the velocity terms in the right-hand side of (4.19) to the left-hand side. Integrating (4.19) in-time,

$$\begin{aligned}&\|\nabla_0 w\|^2_{L^\infty_t L^2_x}+\|\nabla_0\nabla w\|^2_{L^2_t L^2_x}+\|\nabla\eta\|^2_{L^\infty_t L^2_x}+\|\nabla^2\eta\|^2_{L^\infty_t L^2_x}\\&\lesssim \epsilon+\|f\|^2_{L^2_t L^2_x}+\|g\|^2_{L^2_t H^{\frac{1}{2}}_x}.\end{aligned} \tag{4.20}$$

We need to obtain missing terms $\|w_{1,33}\|_{L^2_t L^2_x}$, $\|w_{2,33}\|_{L^2_t L^2_x}$ and $\|w_{3,33}\|_{L^2_t L^2_x}$. Since $\nabla\cdot w=0$, $\|w_{3,33}\|_{L^2}\le 2\|\nabla_0\nabla w\|_{L^2}$. From the equation $w_{i,33}=-w_{i,jj}+w_{i,t}+\partial_i q$, we can replace $\|\nabla_0\nabla w\|_{L^2}$ by $\|\nabla^2 w\|_{L^2}$, by adding $\|w_t\|_{L^2}+\|\partial_i q\|$ to the right-hand side of (4.20). $\nabla q=w_t-\Delta w+f$ implies that

$$\begin{aligned}&\|\partial_3 q\|^2_{L^2_t L^2_x}\lesssim \|w_{3,t}\|^2_{L^2_t L^2_x}+\|\nabla_0\nabla w_3\|^2_{L^2_t L^2_x}+\|f_3\|^2_{L^2_t L^2_x}\\&\lesssim \|w_t\|^2_{L^2_t L^2_x}+\|f\|^2_{L^2_t L^2_x}+\|g\|^2_{L^2_t H^{\frac{1}{2}}_x}+\|\nabla_0 w\|^2_{L^2_t L^2_x}.\end{aligned} \tag{4.21}$$

But, we cannot estimate $\partial_i q$ in terms of $\|\partial_{33} w_i\|_{L^2}$, $i=1,2$ because we cannot bound $\|\partial_{33} w_i\|_{L^2}$ in terms of $\|\nabla_0\nabla w\|_{L^2}$. Therefore, we need to keep the pressure term in the right-hand side of (4.20) such that

$$\begin{aligned}&\|\nabla_0 w\|^2_{L^\infty_t L^2_x}+\|\nabla^2 w\|^2_{L^2_t L^2_x}+\|\nabla\eta\|^2_{L^\infty_t H^1_x}\\&\lesssim \epsilon+\|f\|^2_{L^2_t L^2_x}+\|g\|^2_{L^2_t H^{\frac{1}{2}}_x}+\|w_t\|^2_{L^2_t L^2_x}+\|\nabla q\|^2_{L^2_t L^2_x}.\end{aligned} \tag{4.22}$$

We will obtain $\|w_t\|^2_{L^2_t L^2_x}$ in step 4, and $\|\nabla q\|^2_{L^2_t L^2_x}$ in step 6.

• **Step 3. H^2 estimate:** we take one more derivative to the momentum equation. We multiply the momentum equation by $\Delta_0^2 w$ and integrate over Ω.

$$(w_t,\Delta_0^2 w)+(-\Delta w,\Delta_0^2 w)+(\nabla q,\Delta_0^2 w)=(f,\Delta_0^2 w). \tag{4.23}$$

By the integration by parts,

$$\frac{d}{dt}\|\Delta_0 w\|^2_{L^2}+<\Delta_0 w,\Delta_0 w>+(q-w_{3,3},\Delta_0^2 w_3)\lesssim \|\nabla_0 f\|^2_{L^2}. \tag{4.24}$$

Using the same arguments in step 2,

$$\frac{d}{dt}\Big(\|\Delta_0 w\|^2_{L^2}+\|\nabla^2\eta\|^2_{L^2}+\|\nabla^3\eta\|^2_{L^2}\Big)+\|\nabla(\Delta_0 w)\|^2_{L^2_t L^2_x}\lesssim \|\nabla f\|^2_{L^2}+\|g\|^2_{H^{\frac{3}{2}}}. \tag{4.25}$$

As before, we can replace $\|\nabla(\Delta_0 w)\|^2_{L^2_t L^2_x}$ by $\|\nabla^3 w\|^2_{L^2_t L^2_x}$, by adding $\|\nabla w_t\|^2_{L^2_t L^2_x}+\|\nabla^2 q\|^2_{L^2_t L^2_x}$ to the right-hand side of (4.25). Integrating (4.25) in-time,

$$\begin{aligned}&\|\Delta_0 w\|^2_{L^\infty_t L^2_x}+\|\nabla^3 w\|^2_{L^2_t L^2_x}+\|\nabla^2\eta\|^2_{L^\infty_t H^1_x}\\&\lesssim \epsilon+\|\nabla f\|^2_{L^2_t L^2_x}+\|g\|^2_{L^2_t H^{\frac{3}{2}}_x}+\|\nabla w_t\|^2_{L^2_t L^2_x}+\|\nabla^2 q\|^2_{L^2_t L^2_x}.\end{aligned} \tag{4.26}$$

We will obtain $\|\nabla w_t\|^2_{L^2_t L^2_x}$ in step 5, and $\|\nabla^2 q\|^2_{L^2_t L^2_x}$ in step 6.

• **Step 4. L^2 estimate of w_t:** we need to obtain the energy bound $\|w_t\|^2_{L^2_tL^2_x} + \|\nabla w_t\|^2_{L^2_tL^2_x}$. First, we multiply the momentum equation by w_t and integrate over Ω.

$$(w_t, w_t) + (-\Delta w, w_t) + (\nabla q, w_t) = (f, w_t). \tag{4.27}$$

By the integration by parts,

$$\|w_t\|^2_{L^2} + \frac{1}{2}\frac{d}{dt} < w, w > +(q - w_{3,3}, w_{3,t}) \lesssim \|f\|^2_{L^2}. \tag{4.28}$$

From the boundary condition,

$$\|w_t\|^2_{L^2} + \frac{1}{2}\frac{d}{dt} < w, w > +(\eta - \Delta_0\eta + g_3, w_{3,t}) \lesssim \|f\|^2_{L^2}. \tag{4.29}$$

We estimate boundary terms.

$$(\eta - \Delta_0\eta, \eta_{tt}) = \frac{d}{dt}(\eta - \Delta_0\eta, \eta_t) - \|\eta_t\|^2_{L^2} - \|\nabla\eta_t\|^2_{L^2}. \tag{4.30}$$

By the trace theorem,

$$(g_3, w_{3,t}) \lesssim \|g_3\|^2_{L^2(R^2)} + \frac{1}{4}\|w_{3,t}\|^2_{L^2(R^2)} \lesssim \|g_3\|^2_{L^2} + \frac{1}{4}\|w_{3,t}\|^2_{L^2} + \frac{1}{4}\|\nabla w_{3,t}\|^2_{L^2} \tag{4.31}$$

By (4.30) and (4.31), (4.29) can be written as

$$\begin{aligned} &\|w_t\|^2_{L^2} + \frac{1}{2}\frac{d}{dt} < w, w > +\frac{d}{dt}(\eta - \Delta_0\eta, \eta_t) \\ &\lesssim \|f\|^2_{L^2} + \|g_3\|^2_{L^2} + \frac{1}{4}\|w_{3,t}\|^2_{L^2} + \frac{1}{4}\|\nabla w_{3,t}\|^2_{L^2}. \end{aligned} \tag{4.32}$$

Integrating (4.32) in-time,

$$\begin{aligned} &\|w_t\|^2_{L^2_tL^2_x} + \|\nabla w\|^2_{L^\infty_tL^2_x} \\ \lesssim&\epsilon + \|f\|^2_{L^2_tL^2_x} + \|(\eta - \Delta_0\eta, \eta_t)\|_{L^\infty_tL^2} + \|g\|^2_{L^2_tL^2_x} \\ &+ \|\eta_t\|^2_{L^2_tL^2_x} + \|\nabla\eta_t\|^2_{L^2_tL^2_x} + \|w\|^2_{L^\infty_tL^2_x} + \frac{1}{4}\|\nabla w_{3,t}\|^2_{L^2} \\ \lesssim&\epsilon + \|f\|^2_{L^2_tL^2_x} + \frac{1}{2}\|\eta\|^2_{L^\infty_tH^2_x} + \|\eta_t\|^2_{L^\infty_tL^2_x} + \|g\|^2_{L^2_tH^{\frac{1}{2}}_x} \\ &+ \|\eta_t\|^2_{L^2_tH^1_x} + \|w\|^2_{L^2_tL^2_x} + \frac{1}{4}\|\nabla w_{3,t}\|^2_{L^2}. \end{aligned} \tag{4.33}$$

We need to estimate terms $\|\eta_t\|^2_{L^\infty_tL^2_x} + \|\eta_t\|^2_{L^2_tH^1_x}$ in the right-hand side of (4.33). By the trace theorem and Young's inequality,

$$\|\eta_t\|^2_{L^\infty_tL^2_x} = \|w_3\|^2_{L^\infty_tL^2_x(R^2)} \lesssim \|w\|^2_{L^\infty_tL^2_x} + \frac{1}{2}\|\nabla w\|^2_{L^\infty_tL^2_x},$$

$$\|\eta_t\|^2_{L^2_tL^2_x} = \|w_3\|^2_{L^2_tL^2_x(R^2)} \lesssim \|w\|^2_{L^2_tL^2_x} + \frac{1}{2}\|\nabla w\|^2_{L^2_tL^2_x},$$

$$\|\nabla\eta_t\|^2_{L^2_tL^2_x} = \|\nabla w_3\|^2_{L^2_tL^2_x(R^2)} \lesssim \|w\|^2_{L^2_tL^2_x} + \frac{1}{2}\|\nabla^2 w\|^2_{L^2_tL^2_x}.$$

Therefore, we have the following energy bound:

$$\begin{aligned}
&\|w_t\|^2_{L^2_tL^2_x} + \|\nabla w\|^2_{L^\infty_t L^2_x}\\
\lesssim &\epsilon + \|f\|^2_{L^2_tL^2_x} + \frac{1}{2}\|\eta\|^2_{L^\infty_t H^2_x} + \|g\|^2_{L^2_tH^{\frac{1}{2}}_x} + \frac{1}{2}\|\nabla w\|^2_{L^2_tH^1_x} + \|w\|^2_{L^2_tL^2_x}\\
&+ \frac{1}{4}\|\nabla w_{3,t}\|^2_{L^2} + \|w\|^2_{L^\infty_t L^2_x}\\
\lesssim &\epsilon + \|f\|^2_{L^2_tL^2_x} + \frac{1}{2}\|\eta\|^2_{L^\infty_t H^2_x} + \|g\|^2_{L^2_tH^{\frac{1}{2}}_x} + \frac{1}{2}\|\nabla w\|^2_{L^2_tH^1_x} + \|w\|^2_{L^2_tL^2_x}\\
&+ \frac{1}{2}\|D_h w_t\|^2_{L^2} + \|w\|^2_{L^\infty_t L^2_x},
\end{aligned} \tag{4.34}$$

where we use the divergence-free condition to control $\frac{1}{4}\|\nabla w_{3,t}\|^2_{L^2}$ in terms of the tangential derivatives. By following step 1, we can replace the lower order terms $\|w\|^2_{L^2_tL^2_x} + \|w\|^2_{L^\infty_t L^2_x}$ by forcing terms. Therefore, we have

$$\begin{aligned}
&\|w_t\|^2_{L^2_tL^2_x} + \|\nabla w\|^2_{L^\infty_t L^2_x}\\
&\lesssim \epsilon + \|f\|^2_{L^2_tL^2_x} + \frac{1}{2}\|\eta\|^2_{L^\infty_t H^2_x} + \|g\|^2_{L^2_tH^{\frac{1}{2}}_x} + \frac{1}{2}\|\nabla w\|^2_{L^2_tH^1_x} + \frac{1}{2}\|D_h w_t\|^2_{L^2}.
\end{aligned} \tag{4.35}$$

• **Step 5. H^1 estimate of w_t:** let us take one more derivative. We multiply the momentum equation by $\Delta_0 w_t$. By integrating in the spatial variables,

$$\|\nabla_0 w_t\|^2_{L^2} + \frac{1}{2}\frac{d}{dt} < \nabla_0 w, \nabla_0 w > +(q - w_{3,3}, \Delta_0 w_{3,t}) = (f, \Delta_0 w_t). \tag{4.36}$$

By the same method in step 2 and step 4, we obtain that

$$\begin{aligned}
&\|\nabla_0 w_t\|^2_{L^2_tL^2_x} + \|\nabla_0\nabla w\|^2_{L^\infty_t L^2_x}\\
&\lesssim \epsilon + \|\nabla f\|^2_{L^2_tL^2_x} + \frac{1}{2}\|\nabla\eta\|^2_{L^\infty_t H^2_x} + \|g\|^2_{L^2_tH^{\frac{3}{2}}_x} + \frac{1}{2}\|\nabla^2 w\|^2_{L^2_tH^1_x}.
\end{aligned} \tag{4.37}$$

But, we cannot move to the next equation (4.38) directly from (4.37),

$$\begin{aligned}
&\|\nabla w_t\|^2_{L^2_tL^2_x} + \|\nabla_0\nabla w\|^2_{L^\infty_t L^2_x}\\
&\lesssim \epsilon + \|\nabla f\|^2_{L^2_tL^2_x} + \frac{1}{2}\|\nabla\eta\|^2_{L^\infty_t H^2_x} + \|g\|^2_{L^2_tH^{\frac{3}{2}}_x} + \frac{1}{2}\|\nabla^2 w\|^2_{L^2_tH^1_x}.
\end{aligned} \tag{4.38}$$

To obtain (4.38), we need take the time derivative to the momentum equation.

$$w_{tt} - \Delta w_t + \nabla q_t = f_t. \tag{4.39}$$

By multiplying (4.39) by w_t, and integrating over the domain,

$$\frac{1}{2}\frac{d}{dt}\|w_t\|^2_{L^2} + < w_t, w_t > +(w_{3,3t} + \eta_t - \Delta_0\eta_t + g_{3,t}, w_{3,t}) = (f_t, w_t). \tag{4.40}$$

We can apply Korn's inequality to the second term in the left-hand side of (4.40). Skipping the details, we have the following estimate.

$$\begin{aligned}
&\frac{d}{dt}\left(\|w_t\|^2_{L^2} + \|w_3\|^2_{H^1(R^2)}\right) + \|\nabla w_t\|^2_{L^2}\\
&\lesssim \|f_t\|^2_{L^2} + \|g_t\|^2_{\dot{H}^{-\frac{1}{2}}} + \frac{1}{2}\|w_t\|^2_{L^2} + \frac{1}{2}\|\nabla w_t\|^2_{L^2} + \left|(w_{3,3t}, w_{3,t})\right|\\
&\lesssim \|f_t\|^2_{L^2} + \|g_t\|^2_{\dot{H}^{-\frac{1}{2}}} + \frac{1}{2}\|\nabla w_t\|^2_{L^2} + \left|(w_{3,3t}, w_{3,t})\right|,
\end{aligned} \tag{4.41}$$

where we use Poincare's inequality to control lower order terms by higher order terms. Let us estimate the last term in (4.41). By the divergence-free condition,

$$\Big|(w_{3,3t}, w_{3,t})\Big| \leq \Big|(w_{1,1t}, w_{3,t})\Big| + \Big|(w_{1,1t}, w_{3,t})\Big| \lesssim 2\|\nabla_0 w_t\|_{L^2}^2 + \frac{1}{2}\|\nabla w_t\|_{L^2}^2. \quad (4.42)$$

But, $\|\nabla_0 w_t\|_{L^2}^2$ can be estimated by (4.37). Therefore,

$$\begin{aligned} &\|w_t\|_{L_t^\infty L_x^2}^2 + \|\nabla w_t\|_{L_t^2 L_x^2}^2 + \|\nabla_0 \nabla w\|_{L_t^\infty L_x^2}^2 \\ \lesssim & \epsilon + \|\nabla f\|_{L_t^2 L_x^2}^2 + \|g\|_{L_t^2 H_x^{\frac{3}{2}}}^2 + \|f_t\|_{L_t^2 L_x^2}^2 + \|g_t\|_{L_t^2 \dot{H}_x^{-\frac{1}{2}}}^2 \\ & + \frac{1}{2}\|\nabla \eta\|_{L_t^\infty H_x^2}^2 + \frac{1}{2}\|\nabla^2 w\|_{L_t^2 H_x^1}^2. \end{aligned} \quad (4.43)$$

We will move the last two terms in (4.43) to the left-hand side later.

• **Step 6.** In step 2, and step 3, we have the pressure term in the right-hand side of (4.22) and (4.26). Unfortunately, we cannot derive the estimations of the pressure term from the momentum equation directly because, as we already observed in step 2, we cannot estimate $\|\partial_{33} w_i\|_{L^2}$, $i = 1, 2$ in terms of the tangential derivatives of the velocity field. However, we can derive the energy bounds of full derivatives by solving an elliptic problem with the aid of the estimation of the time derivative of the velocity field. Let us consider the following system of equations:

$$\begin{cases} -\Delta w + \nabla q = F \quad \text{in} \quad \Omega, \quad F = f + w_t \in L_t^2 H_x^1 \\ \nabla \cdot w = 0 \quad \text{in} \quad \Omega, \\ w = 0 \quad \text{on} \quad \{z = -1\}, \\ w_{i,j} + w_{j,i} = g_i \quad \text{on} \quad \{z = 0\}, \end{cases}$$

from which we can show that a weak solution (w, q) satisfies the following estimate:

$$\|w\|_{L_t^2 H_x^3} + \|\nabla q\|_{L_t^2 H_x^1} \lesssim \|f\|_{L_t^2 H_x^1} + \|w_t\|_{L_t^2 H_x^1} + \|g\|_{L_t^2 H_x^{\frac{3}{2}}(R^2)}. \quad (4.44)$$

For the proof, see Lemma 3.3 [3]. In the above system of equations, we have $g_i \neq 0$, $i = 1, 2$, and these terms are easily handled by using argument in Lemma 3.3 [4].

• **Step 7.** In the final step we deal with $\|w_{i,33}\|_{L_t^\infty H_x^1}^2$ by reformulating the momentum equation by projecting it onto the divergence-free space using the same method as in the problem on the moving domain. No details will be provided in this step, and we are only concerned with $\|\mathscr{A} w\|_{L_t^\infty L_x^2}^2$, which is given by

$$\|\mathscr{A} w\|_{L_t^\infty L_x^2}^2 + \int_0^\infty < \mathscr{A} w, \mathscr{A} w > dt \lesssim \epsilon + \|w_t\|_{L_t^2 H_x^1}^2 + \|\nabla q\|_{L_t^2 H_x^1}^2 + \|f\|_{L_t^2 H_x^1}^2 \quad (4.45)$$

By using the same elliptic estimate used in step 6, with the additional term $\|g\|_{L_t^\infty H_x^{\frac{1}{2}}}^2$, we have

$$\|w\|_{L_t^\infty H_x^2} \lesssim \epsilon + \|w_t\|_{L_t^2 H_x^1}^2 + \|\nabla q\|_{L_t^2 H_x^1}^2 + \|f\|_{L_t^2 H_x^1}^2 + \|g\|_{L_t^\infty H_x^{\frac{1}{2}}}^2. \quad (4.46)$$

In sum, collecting all terms from step 1 to step 7, we obtain that

$$\begin{aligned} &\|w\|_{L_t^\infty H_x^2} + \|\nabla w\|_{L_t^2 H_x^2} + \|\eta\|_{L_t^\infty H_x^3} + \|\nabla q\|_{L_t^2 H_x^1} + \|w_t\|_{L_t^2 H_x^1} \\ \lesssim & \|w_0\|_{H^2} + \|\eta_0\|_{H^3} + \|f\|_{L_t^2 H_x^1} + \|f_t\|_{L_t^2 L_x^2} \\ & + \|g\|_{L_t^2 H_x^{\frac{3}{2}}} + \|g_t\|_{L_t^2 \dot{H}_x^{-\frac{1}{2}}} + \|g\|_{L_t^\infty H_x^{\frac{1}{2}}}^2, \end{aligned} \quad (4.47)$$

and this completes the proof of Proposition 4.1. ☐

Remark 2. Under these higher regularity, solutions are unique.

Remark 3. In section 2, we used the vector field method to rewrite the momentum equation to express the pressure term as the harmonic extension of the boundary terms. Now, we take the projection $\mathbb{P}$ to the momentum equation on the equilibrium domain:

$$w_t - \mathbb{P}(\Delta w) + \mathbb{P}(\nabla q) = \mathbb{P}(f).$$

The expression

$$\nabla \mathscr{H}(\eta - \Delta_0 \eta) = \mathbb{P}\nabla q - \nabla \mathscr{H}(w_{3,3}) - \nabla \mathscr{H}(g_3)$$

infers that

$$\|\nabla \mathscr{H}(\eta - \Delta_0 \eta)\|^2_{L^2_t H^1_x} \leq \|\nabla q\|^2_{L^2_t H^1_x} + \|\nabla \mathscr{H}(w_{3,3})\|^2_{L^2_t H^1_x} + \|\nabla \mathscr{H}(g_3)\|^2_{L^2_t H^1_x}.$$

By Proposition 4.1,

$$\|\nabla \mathscr{H}(\eta - \Delta_0 \eta)\|^2_{L^2_t H^1_x} \lesssim \epsilon + \|f\|_{L^2_t H^1_x} + \|g\|_{L^2_t H^{\frac{3}{2}}_x}.$$

Therefore, we can control the boundary in L^2 in-time.

5. **Change of variables.** In this section we present details of the change of variables which is used in section 3. In section 3, we defined a map

$$\theta(t) : \Omega = \{(x_1, x_2, y); -1 < y < 0\} \to \{(x_1, x_2, z^{'}); -1 < z^{'} < \eta(x_1, x_2, t)\},$$

in terms of $\bar{\eta}$, where $\bar{\eta}$ is the harmonic extension of η, such that

$$\theta(x_1, x_2, y, t) = (x_1, x_2, \bar{\eta}(x_1, x_2, t) + y(1 + \bar{\eta}(x_1, x_2, t))).$$

By definition,

$$d\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ A & B & J \end{pmatrix}, \quad \zeta = (d\theta)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{A}{J} & -\frac{B}{J} & \frac{1}{J} \end{pmatrix}$$

$$A = (1+y)\bar{\eta}_{x_1}, \quad B = (1+y)\bar{\eta}_{x_2}, \quad J = 1 + \bar{\eta} + \partial_y \bar{\eta}(1+y)$$

We define v on $\theta(\Omega)$ by $v_i = \dfrac{\theta_{i,j}}{J} w_j = \alpha_{ij} w_j$. Then,

$$v_1 = \frac{w_1}{J}, \quad v_2 = \frac{w_2}{J}, \quad v_3 = \frac{A}{J} w_1 + \frac{B}{J} w_2 + w_3.$$

We make replacements $v_{i,j} = \zeta_{lj} \partial_l (\alpha_{ik} w_k)$.

$$v_{1,1} = \partial_1 \Big(\frac{1}{J} w_1 + \frac{A}{J} w_3 \Big) - \frac{A}{J} \partial_3 \Big(\frac{1}{J} w_1 + \frac{A}{J} w_3 \Big),$$

$$v_{1,2} = \partial_2 \Big(\frac{1}{J} w_1 + \frac{A}{J} w_3 \Big) - \frac{B}{J} \partial_3 \Big(\frac{1}{J} w_1 + \frac{A}{J} w_3 \Big), \quad v_{1,3} = \frac{1}{J} \partial_3 (\frac{1}{J} w_1),$$

$$v_{2,1} = \partial_1 \Big(\frac{1}{J} w_2 + \frac{B}{J} w_3 \Big) - \frac{A}{J} \partial_3 \Big(\frac{1}{J} w_2 + \frac{B}{J} w_3 \Big),$$

$$v_{2,2} = \partial_2 \Big(\frac{1}{J} w_2 + \frac{B}{J} w_3 \Big) - \frac{B}{J} \partial_3 \Big(\frac{1}{J} w_2 + \frac{B}{J} w_3 \Big), \quad v_{2,3} = \frac{1}{J} \partial_3 (\frac{1}{J} w_2),$$

$$v_{3,1} = \partial_1 \Big(\frac{1}{J} A w_1 + \frac{1}{J} B w_2 + w_3 \Big) - \frac{1}{J} A \partial_3 \Big(\frac{1}{J} A w_1 + \frac{1}{J} B w_2 + w_3 \Big),$$

$$v_{3,2} = \partial_2 \Big(\frac{1}{J} A w_1 + \frac{1}{J} B w_2 + w_3 \Big) - \frac{1}{J} B \partial_3 \Big(\frac{1}{J} A w_1 + \frac{1}{J} B w_2 + w_3 \Big),$$

$$v_{3,3} = \frac{1}{J} \partial_3 \Big(\frac{1}{J} A w_1 + \frac{1}{J} B w_2 + w_3 \Big).$$

First, we take time derivative.

$$v_{i,t} = \frac{1}{J}w_{i,t} - \frac{1}{J^2}J_{,t}w_i + ((\theta)^{-1})_3'\Big(\frac{1}{J}w_{i,3} - \frac{1}{J}J_{,3}w_i\Big) \quad \text{for } i = 1,2$$

$$v_{3,t} = \frac{1}{J}Aw_{1,t} - \frac{J_{,t}}{J^2}Aw_1 + \frac{1}{J}A_{,t}w_1 + \frac{1}{J}Bw_{2,t} - \frac{J_{,t}}{J^2}Bw_2 + \frac{1}{J}B_{,t}w_2 + w_{3,t} + ((\theta)^{-1})_3'\Big(- \frac{1}{J^2}J_{,3}Aw_1 + \frac{1}{J}A_{,3}w_1 + \frac{1}{J}Aw_{1,3}\Big) + ((\theta)^{-1})_3'\Big(-\frac{1}{J^2}J_{,3}Bw_2 + \frac{1}{J}B_{,3}w_2 + \frac{1}{J}Bw_{2,3} + w_{3,3}\Big)$$

Next,,we calculate the advection terms.

$$v \cdot \nabla v_i = \frac{1}{J}w_1\partial_1(\frac{1}{J}w_i) + \frac{1}{J}w_2\partial_2(\frac{1}{J}w_i) + \frac{1}{J}w_3\partial_3(\frac{1}{J}w_i) \quad \text{for } i = 1,2$$

$$v \cdot \nabla v_3 = \frac{1}{J}w_1\partial_1\Big(\frac{A}{J}w_1 + \frac{B}{J}w_2 + w_3\Big) + \frac{1}{J}w_2\partial_2\Big(\frac{A}{J}w_1 + \frac{B}{J}w_2 + w_3\Big) + \frac{1}{J}w_3\partial_3\Big(\frac{A}{J}w_1 + \frac{B}{J}w_2 + w_3\Big)$$

Finally, we obtain the dissipation term. We calculate Δ first.

$$\Delta = \partial_{11} + \partial_{22} + \frac{1}{J}\partial_3(\frac{1}{J}\partial_3) - \partial_1(\frac{A}{J}\partial_3) - \partial_2(\frac{B}{J}\partial_3) - \frac{A}{J}\partial_{31} + \frac{A}{J}\partial_3(\frac{1}{J}A\partial_3) - \frac{B}{J}\partial_{32} + \frac{B}{J}\partial_3(\frac{1}{J}B\partial_3)$$

Therefore,

$$\Delta v_i = \partial_{11}(\frac{1}{J}w_i) + \partial_{22}(\frac{1}{J}w_i) + \frac{1}{J}\partial_3(\frac{1}{J}\partial_3(\frac{1}{J}w_i)) - \partial_1(\frac{1}{J}A\partial_3(\frac{1}{J}w_i)) - \partial_2(\frac{1}{J}B\partial_3(\frac{1}{J}w_i)) - \frac{1}{J}A\partial_{31}(\frac{1}{J}w_i) + \frac{1}{J}A\partial_3(\frac{1}{J}A\partial_3(\frac{1}{J}w_i)) - \frac{1}{J}B\partial_{32}(\frac{1}{J}w_i) + \frac{1}{J}B\partial_3(\frac{1}{J}B\partial_3(\frac{1}{J}w_i)) \quad \text{for } i = 1,2$$

$$\Delta v_3 = \partial_{11}\Big(\frac{A}{J}w_1 + \frac{B}{J}w_2 + w_3\Big) + \partial_{22}\Big(\frac{A}{J}w_1 + \frac{B}{J}w_2 + w_3\Big) + \frac{1}{J}\partial_3\Big(\frac{1}{J}\partial_3(\frac{A}{J}w_1 + \frac{B}{J}w_2 + w_3)\Big) - \partial_1\Big(\frac{A}{J}\partial_3(\frac{A}{J}w_1 + \frac{B}{J}w_2 + w_3)\Big) - \partial_2\Big(\frac{B}{J}\partial_3(\frac{A}{J}w_1 + \frac{B}{J}w_2 + w_3)\Big) + \frac{A}{J}\partial_3\Big(\frac{A}{J}\partial_3(\frac{A}{J}w_1 + \frac{B}{J}w_2 + w_3)\Big) - \frac{A}{J}\partial_{31}\Big(\frac{A}{J}w_1 + \frac{B}{J}w_2 + w_3\Big) + \frac{B}{J}\partial_3\Big(\frac{B}{J}\partial_3(\frac{A}{J}w_1 + \frac{B}{J}w_2 + w_3)\Big) - \frac{B}{J}\partial_{32}\Big(\frac{A}{J}w_1 + \frac{B}{J}w_2 + w_3\Big)$$

We define the pressure as $q = p \circ \theta$. Then,

$$\partial_1 p = \partial_1 q - \frac{1}{J}A\partial_3 q, \quad \partial_2 p = \partial_2 q - \frac{1}{J}B\partial_3 q, \quad \partial_3 p = \frac{1}{J}\partial_3 q.$$

We substitute all terms into the Navier-Stokes equations and its boundary conditions. Then, we see the quadratic nonlinear terms mentioned in section 3.

6. Proof of the a priori estimate for the free boundary problems for the Navier-Stokes equations with surface tension.

6.1. Proof of Proposition 2.3: Commutator estimate.

Since the integrand in the right-hand side of (2.18) is cubic, we can estimate terms in $L^2 \times L^4 \times L^4$ or $L^2 \times L^2 \times L^\infty$ by using the Sobolev inequalities, which control L^∞ norm by H^s norm. Moreover, we have to perform the integration by parts to move derivatives to more regular parts, and we therefore need the trace theorem to estimate boundary terms. Finally, we need to estimate $\|\nabla^3 v\|_{L^2}$ in terms of $\|\nabla \mathscr{A} v\|_{L^2}$ to make the a priori estimate closed.

Lemma 6.1. *Sobolev Inequalities* [1]*: for a domain Ω in $\mathbb{R}^3$ with a smooth boundary,*

$$(1)\|f\|_{L^4} \leq C\|f\|_{L^2}^{\frac{1}{4}} \cdot \|\nabla f\|_{L^2}^{\frac{3}{4}}, \quad (2)\|f\|_{L^\infty} \leq C\|f\|_{H^2}.$$

Lemma 6.2. *Trace Theorem:* [1] *let Ω be a domain in R^n having the uniform C^m regularity property and suppose there exists a simple (m,p) extension operator E for Ω. If $mp < n$ and $p \le q \le \frac{(n-1)p}{n-mp}$, then $W^{m,p} \to L^q(\partial\Omega)$.*

Lemma 6.3. *Unique solvability of an elliptic equation: if*

$$\mathscr{A} v = f \quad in \quad \Omega, \quad \hat{t}\cdot T\cdot\hat{n} = 0 \quad on \quad \partial\Omega,$$

then, under the divergence-free condition, $\|v\|_{H^r} \lesssim \|\mathscr{A} v\|_{H^{r-2}}$ for $r = 2, 3$. For the proof, see Lemma 3.3 in [3].

Corollary 6.4. *Sobolev inequalities involving $\mathscr{A} v$.*
(1) $\|\partial v\|_{L^2} \lesssim \|v\|_{L^2}^{\frac{1}{2}} \cdot \|\mathscr{A} v\|_{L^2}^{\frac{1}{2}}$, (2) $\|\mathscr{A} v\|_{L^2} \lesssim \|\partial v\|_{L^2}^{\frac{1}{2}} \cdot \|\nabla(\mathscr{A} v)\|_{L^2}^{\frac{1}{2}}$,
(3) $\|\nabla^3 v\|_{L^2} \lesssim \|\nabla \mathscr{A} v\|_{L^2} + ||\nabla v||_{L^2}$.

Proof. The first inequality is derived from Lemma 6.3 and interpolations,

$$\|\partial v\|_{L^2} \lesssim \|v\|_{L^2}^{\frac{1}{2}} \cdot \|\nabla^2 v\|_{L^2}^{\frac{1}{2}} \lesssim \|v\|_{L^2}^{\frac{1}{2}} \cdot \|\mathscr{A} v\|_{L^2}^{\frac{1}{2}}.$$

The second inequality is obtained by the divergence-free condition of $\mathscr{A} v$:

$$\|\mathscr{A} v\|_{L^2} = (\mathscr{A} v, \mathscr{A} v)^{\frac{1}{2}} =< v, \mathscr{A} v >^{\frac{1}{2}} \le \|\partial v\|_{L^2}^{\frac{1}{2}} \cdot \|\nabla(\mathscr{A} v)\|_{L^2}^{\frac{1}{2}}.$$

The last inequality is obtained by taking $r = 3$ in Lemma 6.3. □

Lemma 6.5. *Korn's inequality: for a velocity field v vanishing at the bottom,*

$$\|v\|_{H^1}^2 \le C < v, v > .$$

For its proof, see Lemma 2.7 in [4].

Now, we prove Proposition 2.3. Since $(\mathscr{A} v \cdot [D_t, \mathscr{A}]v)$ is cubic, we can distribute L_t^∞ and L_t^2 as we want. Therefore, we expect that $\int\int_{\Omega_s} \mathscr{A} v \cdot ([D_t, \mathscr{A}]v) dV dt$ is of the form $(LHS)^2 + \frac{1}{2}(LHS)$, where

$$(LHS) = |||v|||^2 = \Big(\|v\|_{L_t^\infty L_x^2} + \|\mathscr{A} v\|_{L_t^\infty L_x^2} + \|\nabla v\|_{L_t^2 L_x^2} + \|\nabla \mathscr{A} v\|_{L_t^2 L_x^2}\Big)^2. \quad (6.1)$$

If there are no boundary terms, we can obtain the usual commutator estimate,

$$\begin{aligned} &\int_{\Omega_t} [D_t, \mathscr{A}]v \cdot \mathscr{A} v dV \lesssim \int_{\Omega_t} |\nabla v| \cdot |\mathscr{A} v|^2 dx \le \|\nabla v\|_{L^2} \cdot \|\mathscr{A} v\|_{L^4}^2 \\ &\lesssim \|\nabla v\|_{L^2} \cdot \|\mathscr{A} v\|_{L^2}^{\frac{1}{2}} \cdot \|\nabla(\mathscr{A} v)\|_{L^2}^{\frac{3}{2}} \lesssim \|\nabla v\|_{L^2}^4 \cdot \|\mathscr{A} v\|_{L^2}^2 + \frac{1}{2}\|\nabla(\mathscr{A} v)\|_{L^2}^2. \end{aligned} \quad (6.2)$$

Since we have the boundary terms in the operator $\mathscr{A}$, the commutator involves more terms. First, we expand the commutator terms as follows

$$[D_t, \mathscr{A}]v = [D_t, -\mathbb{P}\Delta]v + [D_t, \nabla \mathscr{H}(\hat{n}\cdot T.\cdot \hat{n})]v = (I) + (II).$$

Since ∂_t commutes with $\mathbb{P}\Delta$, (I) can be rewritten as

$$\begin{aligned} I &= [v\cdot\nabla, -\mathbb{P}\Delta]v \\ &= \mathbb{P}\Delta v \cdot \nabla v + \Big(-v\cdot\nabla(\mathbb{P}\Delta v) + \mathbb{P}(v\cdot\nabla\Delta v) + \mathbb{P}(2\nabla v\cdot\nabla\nabla v)\Big) \\ &= (III) + (IV). \end{aligned} \quad (6.3)$$

For (II), we commute terms successively.

$$\begin{aligned}(II) &= D_t(\nabla \mathscr{H}(\hat{n}\cdot T\cdot \hat{n})) - \nabla \mathscr{H}(\hat{n}\cdot T_{D_t v}\cdot \hat{n})\\ &= \nabla D_t \mathscr{H}(\hat{n}\cdot T\cdot \hat{n}) + [D_t,\nabla]\mathscr{H}(\hat{n}\cdot T\cdot \hat{n}) - \nabla \mathscr{H}(\hat{n}\cdot T_{D_t v}\cdot \hat{n})\\ &= \nabla \mathscr{H}(D_t(\hat{n}\cdot T\cdot \hat{n})) + \nabla[D_t,\mathscr{H}](\hat{n}\cdot T\cdot \hat{n}) + [D_t,\nabla]\mathscr{H}(\hat{n}\cdot T\cdot \hat{n})\\ &- \nabla \mathscr{H}(\hat{n}\cdot T_{D_t v}\cdot \hat{n}).\end{aligned} \tag{6.4}$$

The operator D_t on the boundary is understood as

$$D_t(RF) = \Big(\frac{\partial}{\partial t}\{RF\circ u\}\Big)\circ u^{-1} = \frac{\partial}{\partial t}(RF) + v\cdot \frac{\partial}{\partial u}(RF),$$

where RF is the restriction of F onto the free surface S_F and u is the Lagrangian coordinate map solving $\dfrac{dx}{dt} = v(t,x)$, $x(0)=y$. Since $D_t\hat{n}$ is orthogonal to $\hat{n}$,

$$\begin{aligned}&D_t(\hat{n}\cdot T\cdot \hat{n}) = D_t\hat{n}\cdot T\cdot\hat{n} + \hat{n}\cdot D_tT\cdot\hat{n} + \hat{n}\cdot T\cdot D_t\hat{n}\\ =&\hat{n}\cdot D_tT\cdot\hat{n} = n_in_jD_tR(v_{i,j}+v_{j,i})\\ =&n_in_jR(D_t(v_{i,j}+v_{j,i})) + n_in_j[\partial_t,R](v_{i,j}+v_{j,i}) + n_in_j[v\cdot\nabla,R](v_{i,j}+v_{j,i})\\ =&n_in_j\Big(((D_tv)_{i,j}+(D_tv)_{j,i}) + ([D_t,\partial_j]v_i + [D_t,\partial_i]v_j)\\ +&\partial_t\eta R(\partial_z(v_{i,j}+v_{j,i})) + [v\cdot\nabla,R](v_{i,j}+v_{j,i})\Big)\\ =&\hat{n}\cdot T_{D_tv}\cdot\hat{n} + n_in_j([D_t,\partial_j]v_i + [D_t,\partial_i]v_j) + n_in_j\partial_t\eta R(\partial_z(v_{i,j}+v_{j,i}))\\ +&n_in_j\Big(Rv_i\partial_i\eta R(\partial_z(v_{i,j}+v_{j,i})) + Rv_3R(\partial_z(v_{i,j}+v_{j,i}))\Big).\end{aligned} \tag{6.5}$$

After reordering terms in (6.4),

$$\begin{aligned}(II) &= \nabla\mathscr{H}\Big(n_in_j([D_t,\partial_j]v_i\\ &+ [D_t,\partial_i]v_j)\Big) + \nabla[D_t,\mathscr{H}](\hat{n}\cdot T\cdot\hat{n}) + [D_t,\nabla]\mathscr{H}(\hat{n}\cdot T\cdot\hat{n})\\ &+ \nabla\mathscr{H}\Big(n_in_j\partial_t\eta R(\partial_z(v_{i,j}+v_{j,i})) + n_in_j\big\{Rv_i\partial_i\eta R(\partial_z(v_{i,j}+v_{j,i}))\\ &- Rv_3R(\partial_z(v_{i,j}+v_{j,i}))\big\}\Big)\\ &= (V)+(VI)+(VII)+(VIII).\end{aligned} \tag{6.6}$$

By the identity used in Shatah-Zeng [14],

$$(VI) = \nabla(\Delta)^{-1}\Big(2\partial v\cdot\nabla^2\mathscr{H}(\hat{n}\cdot T\cdot\hat{n}) + \nabla\mathscr{H}(\hat{n}\cdot T\cdot\hat{n})\cdot\Delta v\Big). \tag{6.7}$$

Here, Δ^{-1} denotes the inverse of the Laplacian with zero Dirichlet boundary condition at the free surface. Therefore, (VI) is orthogonal to $\mathscr{A}v$. Finally,

$$(VII) = -\nabla v\cdot\nabla\mathscr{H}(n\cdot T\cdot n), \tag{6.8}$$

where $(\cdot)$ is the matrix multiplication with a vector, not the inner product. By adding (III) and (VII),

$$(III)+(VII) = -\nabla v\cdot\Big(-\mathbb{P}\Delta v + \nabla\mathscr{H}(n\cdot T\cdot n)\Big) = -\nabla v\cdot\mathscr{A}v. \tag{6.9}$$

- **Estimation of (III)+(VII):** it is the same as (6.2).

$$\int_{\Omega_t}\big((III)+(VII)\big)\cdot\mathscr{A}v\,dV \lesssim \|\nabla v\|_{L^2}^4\cdot\|\mathscr{A}v\|_{L^2}^2 + \frac{1}{2}\|\nabla(\mathscr{A}v)\|_{L^2}^2. \tag{6.10}$$

• **Estimation of (IV):** since $\mathbb{P}$ is a bounded operator in H^1,

$$\begin{aligned}
&\int_{\Omega_t} \mathscr{A} v \cdot \Big(- v\cdot\nabla(\mathbb{P}\Delta v) + \mathbb{P}(v\cdot\nabla\Delta v) + \mathbb{P}(2\nabla v\cdot\nabla\nabla v)\Big) dV \\
&\lesssim \int_{\Omega_t} |\mathscr{A} v| \Big(|v\cdot\nabla(\mathbb{P}\Delta v)| + |\mathbb{P}(v\cdot\nabla\Delta v)| + |\mathbb{P}(2\nabla v\cdot\nabla\nabla v)|\Big) dV \\
&\lesssim \|\mathscr{A} v\|_{L^2}\|\nabla v\nabla^2 v\|_{L^2} + \|\mathscr{A} v\|_{L^2}\|v\|_{L^\infty}\Big(\|\nabla^3 v\|_{L^2} + \|\nabla^2 v\|_{L^2}\Big) \\
&\lesssim \|\mathscr{A} v\|_{L^2}\cdot\|\nabla v\|_{L^4}\cdot\|\nabla^2 v\|_{L^4} \\
&\quad + \|\mathscr{A} v\|_{L^2}\Big(\|v\|_{L^2} + \|\mathscr{A} v\|_{L^2}\Big)\Big(\|\nabla\mathscr{A} v\|_{L^2} + \|\nabla v\|_{L^2} + \|\mathscr{A} v\|_{L^2}\Big) \\
&\lesssim \|\mathscr{A} v\|_{L^2}^2\|\nabla v\|_{L^2}^{\frac{1}{2}}\|\nabla^3 v\|_{L^2}^{\frac{3}{4}} + \|v\|_{L^2}^2\|\mathscr{A} v\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|\mathscr{A} v\|_{L^2}\|\nabla v\|_{L^2}^2 \\
&\lesssim \|\mathscr{A} v\|_{L^2}^2\|\nabla v\|_{L^2} + \|\mathscr{A} v\|_{L^2}^2\|\nabla v\|_{L^2}^{\frac{1}{4}}\|\nabla\mathscr{A} v\|_{L^2}^{\frac{3}{4}} + \|v\|_{L^2}^2\|\mathscr{A} v\|_{L^2}^2 \\
&\quad + \|\nabla v\|_{L^2}^2 + \|\mathscr{A} v\|_{L^2}\|\nabla v\|_{L^2}^2 \\
&\lesssim \frac{1}{2}\|\nabla\mathscr{A} v\|_{L^2}^2 + \|\nabla v\|_{L^2}^4 + \|\mathscr{A} v\|_{L^2}^{\frac{16}{5}}\|\nabla v\|_{L^2}^{\frac{2}{5}} + \|v\|_{L^2}^2\|\mathscr{A} v\|_{L^2}^2 \\
&\quad + \|\nabla v\|_{L^2}^2 + \|\mathscr{A} v\|_{L^2}\|\nabla v\|_{L^2}^2.
\end{aligned} \tag{6.11}$$

• **Estimation of (V):** since

$$\begin{aligned}
(V) =& \nabla\mathscr{H}\Big(n_i n_j([D_t,\partial_j]v_i + [D_t,\partial_i]v_j)\Big) \\
& -\nabla\mathscr{H}\Big(n_i n_j(\partial_j v\cdot\nabla v_i + \partial_i v\cdot\nabla v_j)\Big),
\end{aligned} \tag{6.12}$$

we have

$$\begin{aligned}
&\int_{\Omega_t} \mathscr{A} v \cdot \Big(-\nabla\mathscr{H}(n_i n_j(\partial_j v\cdot\nabla v_i + \partial_i v\cdot\nabla v_j))\Big) dV \\
&\lesssim \|\hat{n}\cdot\mathscr{A} v\|_{L^2(\partial_{\Omega_t})}\|\Big(n_i n_j(\partial_j v\cdot\nabla v_i + \partial_i\cdot\nabla v_j)\Big)\|_{L^2(\partial\Omega_t)} \\
&\lesssim \|\nabla(\mathscr{A} v)\|_{L^2}\|\nabla v\cdot\partial v\|_{H^1} + \|\mathscr{A} v\|_{L^2}\|\nabla v\cdot\nabla v\|_{H^1} = ⓐ + ⓑ.
\end{aligned} \tag{6.13}$$

$$\begin{aligned}
&ⓐ \lesssim \|\nabla(\mathscr{A} v)\|_{L^2}\Big(\|\nabla v\cdot\nabla v\|_{L^2} + \|\nabla(\nabla v\cdot\nabla v)\|_{L^2}\Big) \\
&\lesssim \|\nabla(\mathscr{A} v)\|_{L^2}\|\nabla v\|_{L^2}\|\nabla v\|_{L^\infty} + \|\nabla(\mathscr{A} v)\|_{L^2}\|\nabla^2 v\|_{L^4}\|\nabla v\|_{L^4} \\
&\lesssim \|\nabla\mathscr{A} v\|_{L^2}\|\nabla v\|_{L^2}\Big(\|\nabla v\|_{L^2} + \|\nabla^3 v\|_{L^2}\Big) \\
&\quad + \|\nabla\mathscr{A} v\|_{L^2}\|\nabla^2 v\|_{L^2}\|\nabla v\|_{L^2}^{\frac{1}{4}}\|\nabla^3 v\|_{L^2}^{\frac{3}{4}} \\
&\lesssim \|\nabla\mathscr{A} v\|_{L^2}\|\nabla v\|_{L^2}^2 + \|\nabla\mathscr{A} v\|_{L^2}^2\|\nabla v\|_{L^2} + \|\nabla\mathscr{A} v\|_{L^2}^{\frac{7}{4}}\|\nabla^2 v\|_{L^2}\|\nabla v\|_{L^2}^{\frac{1}{4}} \\
&\lesssim \frac{1}{2}\|\nabla\mathscr{A} v\|_{L^2}^2 + \|\nabla v\|_{L^2}^4 + \|\nabla\mathscr{A} v\|_{L^2}^2\|\nabla v\|_{L^2} \\
&\quad + \|\nabla^2 v\|_{L^2}^8\|\nabla v\|_{L^2}^2 + \|\nabla^2 v\|_{L^2}^2\|\nabla v\|_{L^2}^2 \\
&\lesssim \frac{1}{2}\|\nabla\mathscr{A} v\|_{L^2}^2 + \|\nabla v\|_{L^2}^4 + \|\nabla\mathscr{A} v\|_{L^2}^2\|\nabla v\|_{L^2} \\
&\quad + \|\mathscr{A} v\|_{L^2}^8\|\nabla v\|_{L^2}^2 + \|\mathscr{A} v\|_{L^2}^2\|\nabla v\|_{L^2}^2.
\end{aligned} \tag{6.14}$$

$$
\begin{aligned}
&ⓑ \lesssim \|\mathscr{A}v\|_{L^2}\Big(\|\nabla v\|_{H^1}\cdot\|\nabla v\|_{L^\infty}\Big)\\
\lesssim&\|\mathscr{A}v\|_{L^2}\|\nabla v\|_{L^2}\Big(\|\nabla^3 v\|_{L^2}+\|\nabla v\|_{L^2}\Big)\\
&+\|\mathscr{A}v\|_{L^2}\|\nabla^2 v\|_{L^2}\Big(\|\nabla^3 v\|_{L^2}+\|\nabla v\|_{L^2}\Big)\\
\lesssim&\|\mathscr{A}v\|_{L^2}\|\nabla v\|_{L^2}\|\nabla\mathscr{A}v\|_{L^2}+\|\mathscr{A}v\|_{L^2}\|\nabla v\|_{L^2}^2\\
&+\|\mathscr{A}v\|_{L^2}\|\nabla^2 v\|_{L^2}\|\nabla\mathscr{A}v\|_{L^2}\\
\lesssim&\frac{1}{2}\|\nabla\mathscr{A}v\|_{L^2}^2+\|\mathscr{A}v\|_{L^2}^2\Big(\|\mathscr{A}v\|_{L^2}+\|\nabla v\|_{L^2}\Big)^2+\Big(\|\mathscr{A}v\|_{L^2}+\|\nabla v\|_{L^2}\Big)^3.
\end{aligned}
\tag{6.15}
$$

• **Estimation of (VIII):** (VIII) is given by

$$
\begin{aligned}
(VIII)=&\nabla\mathscr{H}\Big(n_in_j\partial_t\eta R(\partial_z(v_{i,j}+v_{j,i}))+n_in_j\big\{Rv_i\partial_i\eta R(\partial_z(v_{i,j}+v_{j,i}))\\
&\qquad -Rv_3R(\partial_z(v_{i,j}+v_{j,i}))\big\}\Big).
\end{aligned}
\tag{6.16}
$$

Therefore,

$$
\begin{aligned}
&\int_{\Omega_t}\mathscr{A}v\cdot(VIII)dV=\int_{\partial\Omega_t}(\hat{n}\cdot\mathscr{A}v)\Big(n_in_j\eta_tR(\partial_z(v_{i,j}+v_{j,i}))\Big)dS\\
&\quad+\int_{\partial\Omega_t}(\hat{n}\cdot\mathscr{A}v)\Big(n_in_jRv_i\partial_i\eta R(\partial_z(v_{i,j}+v_{j,i}))\Big)dS\\
&\quad-\int_{\partial\Omega_t}(\hat{n}\cdot\mathscr{A}v)\Big(n_in_jRv_3R(\partial_z(v_{i,j}+v_{j,i}))\Big)dS\\
\lesssim&\|\mathscr{A}v\|_{H^1}\|v\cdot\nabla^2 v\|_{H^1}\\
\lesssim&(\|\mathscr{A}v\|_{L^2}+\|\nabla\mathscr{A}v\|_{L^2})\Big(\|v\cdot\nabla^2v\|_{L^2}+\|\nabla(v\cdot\nabla^2v)\|_{L^2}\Big)\\
\lesssim&(\|\mathscr{A}v\|_{L^2}+\|\nabla\mathscr{A}v\|_{L^2})\Big(\|v\cdot\nabla^2v\|_{L^2}+\|\nabla v\cdot\nabla^2v\|_{L^2}+\|v\cdot\nabla^3v\|_{L^2}\Big)\\
=&\|\mathscr{A}v\|_{L^2}\Big(\|v\cdot\nabla^2v\|_{L^2}+\|\nabla v\cdot\nabla^2v\|_{L^2}+\|v\cdot\nabla^3v\|_{L^2}\Big)\\
&\quad+\|\nabla\mathscr{A}v\|_{L^2}\Big(\|v\cdot\nabla^2v\|_{L^2}+\|\nabla v\cdot\nabla^2v\|_{L^2}+\|v\cdot\nabla^3v\|_{L^2}\Big)=ⓒ+ⓓ.
\end{aligned}
\tag{6.17}
$$

$$
\begin{aligned}
&ⓒ\lesssim\|\mathscr{A}v\|_{L^2}^2\|v\|_{L^\infty}+\|\mathscr{A}v\|_{L^2}^2\|\nabla v\|_{L^\infty}+\|\mathscr{A}v\|_{L^2}\|v\|_{L^\infty}\|\nabla^3v\|_{L^2}\\
\lesssim&\|\mathscr{A}v\|_{L^2}^2\|v\|_{L^2}+\|\mathscr{A}v\|_{L^2}^3+\|\mathscr{A}v\|_{L^2}^2\|\nabla v\|_{L^2}+\|\mathscr{A}v\|_{L^2}^2\|\nabla^3v\|_{L^2}\\
&\quad+\|\mathscr{A}v\|_{L^2}\Big(\|v\|_{L^2}+\|\mathscr{A}v\|_{L^2}\Big)\Big(\|\nabla v\|_{L^2}+\|\nabla\mathscr{A}v\|_{L^2}\Big)\\
\lesssim&\|\mathscr{A}v\|_{L^2}^2\Big(\|v\|_{L^2}+\|\mathscr{A}v\|_{L^2}+\|\nabla v\|_{L^2}+\|\nabla\mathscr{A}v\|_{L^2}\Big)\\
&\quad+\|\mathscr{A}v\|_{L^2}\|v\|_{L^2}\Big(\|\nabla v\|_{L^2}+\|\nabla\mathscr{A}v\|_{L^2}\Big)\\
\lesssim&\|\nabla v\|_{L^2}\|\nabla\mathscr{A}v\|_{L^2}\Big(\|\mathscr{A}v\|_{L^2}+\|v\|_{L^2}+\|\nabla v\|_{L^2}+\|\nabla\mathscr{A}v\|_{L^2}^2+\|v\|_{L^2}^2\Big)\\
&\quad+\|\nabla v\|_{L^2}^{\frac{3}{2}}\|v\|_{L^2}\|\nabla\mathscr{A}v\|_{L^2}^{\frac{1}{2}}+\frac{1}{2}\|\nabla\mathscr{A}v\|_{L^2}^2\\
\lesssim&\|v\|_{L^2}^2\|\nabla v\|_{L^2}^2+\|\mathscr{A}v\|_{L^2}^2\|\nabla v\|_{L^2}^2+\|\nabla v\|_{L^2}^4+\|\nabla v\|_{L^2}\|\nabla\mathscr{A}v\|_{L^2}^2\\
&\quad+\|v\|_{L^2}^{\frac{4}{5}}\|\nabla v\|_{L^2}^2+\|v\|_{L^2}^4\|\nabla v\|_{L^2}^2+\frac{1}{2}\|\nabla\mathscr{A}v\|_{L^2}^2.
\end{aligned}
\tag{6.18}
$$

$$
\begin{aligned}
ⓓ &\lesssim \|\nabla \mathscr{A} v\|_{L^2}\|v\|_{L^\infty}\|\nabla^2 v\|_{L^2} + \|\nabla \mathscr{A} v\|_{L^2}\|\nabla v\|_{L^\infty}\|\nabla^2 v\|_{L^2} \\
&\quad + \|\nabla \mathscr{A} v\|_{L^2}\|v\|_{L^\infty}\|\nabla^3 v\|_{L^2} \\
&\lesssim \|\nabla \mathscr{A} v\|_{L^2}\Big(\|v\|_{L^2} + \|\mathscr{A} v\|_{L^2} + \|\nabla v\|_{L^2} + \|\nabla^3 v\|_{L^2}\Big)\|\mathscr{A} v\|_{L^2} \\
&\quad + \|\nabla \mathscr{A} v\|_{L^2}\Big(\|v\|_{L^2} + \|\mathscr{A} v\|_{L^2}\Big)\Big(\|\nabla v\|_{L^2} + \|\nabla \mathscr{A} v\|_{L^2}\Big) \\
&\lesssim \|v\|_{L^2}^2\|\nabla v\|_{L^2}^2 + \|\nabla v\|_{L^2}^4 + \|\mathscr{A} v\|_{L^2}^2\|\nabla v\|_{L^2}^2 + \|v\|_{L^2}^2\|\nabla v\|_{L^2}^2 \\
&\quad + \Big(\|v\|_{L^2} + \|\mathscr{A} v\|_{L^2}\Big)\|\nabla \mathscr{A} v\|_{L^2}^2 + \frac{1}{2}\|\nabla \mathscr{A} v\|_{L^2}^2.
\end{aligned} \tag{6.19}
$$

Collecting all terms, we finish the proof of Proposition 2.3.

6.2. **Proof of Proposition 2.4, 2.5, and 2.6.** In this section we study the term (α). First, we write all terms in (α).

$$
\begin{aligned}
(\alpha) &= \int_{\partial\Omega_t} (\hat{n}\cdot \mathscr{A} v)\Big(n_k n_l \partial_k \partial_l \mathscr{H}(\eta - \Delta_0 \eta)\Big) dS \\
&+ \int_{\partial\Omega_t} \frac{1}{\sqrt{1+|\nabla\eta|^2}} \Delta_0(v\cdot\nabla\eta)\cdot \Delta_0(\eta - \Delta_0\eta) dS \\
&+ \int_{\partial\Omega_t} \Big(\frac{|\nabla\eta|^2}{1+|\nabla\eta|^2}(\hat{n}\cdot\mathscr{A} v) + \frac{\partial_1\eta(\mathscr{A} v)_1 + \partial_2(\mathscr{A} v)_2}{\sqrt{1+|\nabla\eta|^2}}\Big)\Delta_0\mathscr{H}(\eta - \Delta_0\eta) dS \\
&- \int_{\partial\Omega_t} \frac{\Delta_0 v_3}{\sqrt{1+|\nabla\eta|^2}}\Big\{2\nabla\eta\cdot\nabla\partial_3\mathscr{H}(\eta-\Delta_0\eta) + \Delta_0\eta\partial_3\mathscr{H}(\eta-\Delta_0\eta) \\
&+ \frac{1}{2}|\nabla\eta|^2\partial_3\partial_3\mathscr{H}(\eta-\Delta_0\eta)\Big\} dS \\
&- \int_{\partial\Omega_t} \frac{\Delta_0\mathscr{H}(\eta-\Delta_0\eta)}{\sqrt{1+|\nabla\eta|^2}}\Big\{\big(2\nabla\eta\cdot\nabla\partial_3 v_3 + \Delta_0\eta\partial_3 v_3 + \frac{1}{2}|\nabla\eta|^2\partial_3\partial_3 v_3\big) \\
&- \big(\partial_3\partial_3 v_3 - \partial_3\mathscr{H}(\hat{n}\cdot T\cdot\hat{n})\big)\Big\} dS \\
&+ \int_{\partial\Omega_t} (n\cdot\mathscr{A} v)\Big(n_i n_j \partial_i\partial_j\mathscr{H}(\Delta_0\eta - F(\eta))\Big) dS,
\end{aligned} \tag{6.20}
$$

where the first integral is summed up over $k, l = 1, 2, 3$ except for $k = l = 3$. We have to show that $(\partial_3\partial_3 v_3 - \partial_3\mathscr{H}(n\cdot T\cdot n))$ is quadratic.

$$
\begin{aligned}
\partial_3\partial_3 v_3 - \partial_3\mathscr{H}(n\cdot T\cdot n) &= \Big(\partial_3\partial_3 v_3 - \partial_3\mathscr{H}(\partial_3 v_3)\Big) \\
&+ \partial_3\mathscr{H}\Big(\frac{|\nabla\eta|^2}{1+|\nabla\eta|^2}\partial_3 v_3\Big) - \partial_3\mathscr{H}\Big(n_k n_l(v_{k,l} + v_{l,k})\Big).
\end{aligned} \tag{6.21}
$$

Since $R\mathscr{H} = Id$, $(\partial_3\partial_3 v_3 - \partial_3\mathscr{H}(\partial_3 v_3)) = 0$ and $(\partial_3\partial_3 v_3 - \partial_3\mathscr{H}(n\cdot T\cdot n))$ is quadratic with coefficients $\nabla\eta$. We have half more derivative to the harmonic extension parts and half less derivative to the velocity field parts. Since all terms only depend on (x, y), we transform $\partial\Omega_t$ to R^2, move half derivative from harmonic extension parts to the velocity field parts. When we transform $\partial\Omega_t$ to R^2 and vice versa, the factor $\sqrt{1+|\nabla\eta|^2}$ and its reciprocal appear, but these terms do not change the estimations below.

$$
\begin{aligned}
(\alpha) \lesssim & \|\eta\|_{H_x^3}\Big(\|\nabla v\|_{L^2}^2 + \|\nabla \mathscr{A} v\|_{L^2}^2\Big) + \frac{1}{2}\|\nabla \mathscr{H}(\eta - \Delta_0\eta)\|_{H^1}^2 \\
& + \int_{\partial\Omega_t} (n\cdot \mathscr{A} v)\Big(n_i n_j \partial_i\partial_j \mathscr{H}(\Delta_0\eta - F(\eta))\Big) dS \\
\lesssim & \|\eta\|_{H_x^3}\Big(\|\nabla v\|_{L^2}^2 + \|\nabla \mathscr{A} v\|_{L^2}^2\Big) + \frac{1}{2}\|\nabla \mathscr{H}(\eta - \Delta_0\eta)\|_{H^1}^2 \\
& + \|\eta\|_{H_x^3}\|\mathscr{A} v\|_{H^1}^2 + \Big(\frac{1}{\|\eta\|_{H_x^3}}\|\nabla\eta\|_{L^\infty}^2\Big)\|\nabla\eta\|_{L^\infty}^2\|\nabla^{\frac{5}{2}}\eta\|_{H^1}^2 \\
\lesssim & \|\eta\|_{H_x^3}\Big(\|\nabla v\|_{L^2}^2 + \|\nabla \mathscr{A} v\|_{L^2}^2\Big) + \frac{1}{2}\|\nabla \mathscr{H}(\eta - \Delta_0\eta)\|_{H^1}^2 \\
& + \|\eta\|_{H^3}^2\|\nabla^{\frac{5}{2}}\eta\|_{H^1}^2.
\end{aligned}
\tag{6.22}
$$

Therefore, integrating (6.22) in-time,

$$
\begin{aligned}
\int(\alpha)dt \lesssim & \|\eta\|_{L_t^\infty H_x^3}\Big(\|\nabla v\|_{L_t^2L_x^2}^2 + \|\nabla \mathscr{A} v\|_{L_t^2L_x^2}^2\Big) + \|\eta\|_{L_t^\infty H_x^3}^2\|\nabla^{\frac{5}{2}}\eta\|_{L_t^2H_x^1}^2 \\
& + \frac{1}{2}\|\nabla \mathscr{H}(\eta - F(\eta))\|_{L_t^2H_x^1}^2.
\end{aligned}
\tag{6.23}
$$

We have to estimate the term $\frac{1}{2}\|\nabla \mathscr{H}(\eta - F(\eta))\|_{L_t^2H_x^1}^2$ in (6.23). From the momentum equation,

$$
\|\nabla \mathscr{H}(\eta - F(\eta))\|_{L_t^2H_x^1}^2 \lesssim \|v_t\|_{L_t^2H_x^1}^2 + |||v|||^4 + \|v\|^2. \tag{6.24}
$$

Here, we need the factor $\dfrac{1}{2}$ in (6.23) to move terms in the right-hand side of the following lemma to the left hand of (2.17) in section 2.

Lemma 6.6. *v_t satisfies the following estimate:*

$$
\|v_t\|_{L_t^2H_x^1}^2 \lesssim \epsilon + |||v|||^4 + |||v|||^2 + |||v|||^2\|\eta\|_{L_t^\infty H_x^3}^2 + |||v|||^2\|v_t\|_{L_t^2H_x^1}^2. \tag{6.25}
$$

Proof. Since $v_t = 0$ at the bottom,

$$
\|v_t\|_{L_t^2H_x^1} \lesssim \int < v_t, v_t > dt.
$$

To $\int < v_t, v_t > dt$, we take D_t to the momentum equation.

$$
\begin{aligned}
& D_t\Big(v_t + \mathbb{P}(v\cdot\nabla v)\Big) + \mathscr{A}(v_t + \mathbb{P}(v\cdot\nabla v)) + D_t\nabla\mathscr{H}(\eta - F(\eta)) \\
& - [D_t v, \mathscr{A}]v - \mathscr{A}(v\cdot\nabla v - \mathbb{P}(v\cdot\nabla v)).
\end{aligned}
\tag{6.26}
$$

We multiply (6.26) by $(v_t + \mathbb{P}(v\cdot\nabla v))$ and integrate over Ω_t.

$$
\begin{aligned}
& \frac{d}{dt}\|v_t + \mathbb{P}(v\cdot\nabla v)\|_{L^2}^2 + < v_t + \mathbb{P}(v\cdot\nabla v), v_t + \mathbb{P}(v\cdot\nabla v) > \\
\lesssim & \|v_t + \mathbb{P}(v\cdot\nabla v)\|_{L^2}^2 + \int [D_t, \mathscr{A}]v\cdot(v_t + \mathbb{P}(v\cdot\nabla v))dV + \|\partial^2(v\cdot\nabla v)\|_{L^2}^2 \\
& + \Big(D_t\nabla\mathscr{H}(\eta - F(\eta)), v_t + \mathbb{P}(v\cdot\nabla v)\Big).
\end{aligned}
\tag{6.27}
$$

Now, we calculate $D_t\nabla\mathscr{H}(\eta - F(\eta))$ by commuting operators successively.

$$
\begin{aligned}
D_t\nabla\mathscr{H}(\eta - F(\eta)) = & \nabla\mathscr{H}(D_t(\eta - F(\eta))) + \nabla(\Delta)^{-1}\Delta(v\cdot\nabla\mathscr{H}(\eta - F(\eta))) \\
& - \nabla v\cdot\nabla\mathscr{H}(\eta - F(\eta)).
\end{aligned}
\tag{6.28}
$$

As before, Δ^{-1} denotes the inverse of the Laplacian with zero Dirichlet boundary condition at the free surface. Therefore, $([D_t,\nabla]\mathscr{H}(\eta-F(\eta)))$ is orthogonal to $(v_t+\mathbb{P}(v\cdot\nabla v))$.

$$\begin{aligned}&\frac{d}{dt}\|v_t+\mathbb{P}(v\cdot\nabla v)\|_{L^2}^2+<v_t+\mathbb{P}(v\cdot\nabla v),v_t+\mathbb{P}(v\cdot\nabla v)>\\ \lesssim&\|v_t+\mathbb{P}(v\cdot\nabla v)\|_{L^2}^2+\int_{\Omega_t}[D_t,\mathscr{A}]v\cdot(v_t+\mathbb{P}(v\cdot\nabla v))dV\\ &+\|\mathscr{A}(v\cdot\nabla v-\mathbb{P}(v\cdot\nabla v))\|_{L^2}^2+\|\nabla\mathscr{H}(\partial_t(\eta-F(\eta)))\|_{L^2}^2\\ &+\|\nabla\mathscr{H}(v\cdot\nabla(\eta-F(\eta)))\|_{L^2}^2+\|\nabla v\cdot\nabla\mathscr{H}(\eta-F(\eta))\|_{L^2}^2\end{aligned}\tag{6.29}$$

We can estimate $\|v_t+\mathbb{P}(v\cdot\nabla v)\|_{L^2}^2$ by $\|v_t\|_{L^2}^2+|||v|||^4$ and we can control $\|\mathscr{A}(v\cdot\nabla v-\mathbb{P}(v\cdot\nabla v))\|_{L^2}^2$ by $|||v|||^4$. Remaining terms are

$$\begin{aligned}&\|\nabla\mathscr{H}(v\cdot\nabla(\eta-F(\eta)))\|_{L^2}^2+\|\nabla(\Delta)^{-1}\Delta(v\cdot\nabla\mathscr{H}(\eta-F(\eta)))\|_{L^2}^2\\ &+\|\nabla\mathscr{H}(\partial_t(\eta-F(\eta)))\|_{L^2}^2\\ \lesssim&|||v|||^2\cdot\|\nabla\mathscr{H}(\eta-F(\eta))\|_{H^1}^2+\|\eta_t\|_{L^2}^2+\|\nabla^{\frac{5}{2}}\eta_t\|_{L^2}^2.\end{aligned}\tag{6.30}$$

Integrating (6.29) in-time,

$$\begin{aligned}&\int<v_t+\mathbb{P}(v\cdot\nabla v),v_t+\mathbb{P}(v\cdot\nabla v)>dt\\ \lesssim&\epsilon+\frac{1}{2}\|v_t\|_{L_t^2L_x^2}^2+|||v|||^4+|||v|||^2\|\nabla\mathscr{H}(\eta-F(\eta))\|_{H^1}^2\\ &+\|\eta_t\|_{L_t^2L_x^2}^2+\|\nabla^{\frac{5}{2}}\eta_t\|_{L_t^2L_x^2}^2+\int\int_{\Omega_t}[D_tv,\mathscr{A}]v\cdot(v_t+\mathbb{P}(v\cdot\nabla v))dVdt\\ \lesssim&\epsilon+\frac{1}{2}\|v_t\|_{L_t^2L_x^2}^2+|||v|||^4+\frac{1}{2}\|\nabla\mathscr{H}(\eta-F(\eta))\|_{L_t^2H_x^1}^2+\|\eta\|_{L_t^\infty H_x^3}^2|||v|||^2\\ &+|||v|||^2+|||v|||^2\cdot\|\nabla^{\frac{5}{2}}\eta\|_{L_t^2H_x^1}^2\\ &+\int\int_{\Omega_t}[D_tv,\mathscr{A}]v\cdot(v_t+\mathbb{P}(v\cdot\nabla v))dVdt.\end{aligned}\tag{6.31}$$

Finally, we need to estimate the term: $\int\int_{\Omega_t}[D_tv,\mathscr{A}]v\cdot(v_t+\mathbb{P}(v\cdot\nabla v))dVdt$. By the divergence-free condition of $v_t+\mathbb{P}(v\cdot\nabla v)$, we do the same estimate by replacing $\mathscr{A}v$ with $v_t+\mathbb{P}(v\cdot\nabla v)$ in the proof of Proposition 2.3. Up to signs and (n_in_j),

$$\begin{aligned}&\int[D_tv,\mathscr{A}]v\cdot(v_t+\mathbb{P}(v\cdot\nabla v))dV=\int(\nabla v\cdot\mathscr{A}v)\cdot(v_t+\mathbb{P}(v\cdot\nabla v))dV\\ &+\int\Big(\mathbb{P}(\nabla v\cdot\nabla^2v)+\mathbb{P}(v\cdot\nabla\Delta v)+v\cdot\nabla(\mathbb{P}(\Delta v))\Big)\cdot\Big(v_t+\mathbb{P}(v\cdot\nabla v)\Big)dV\\ &+\int_{\partial\Omega_t}n\cdot\Big(v_t+\mathbb{P}(v\cdot\nabla v)\Big)(\nabla v)^2dS\\ &+\int_{\partial\Omega_t}n\cdot\Big(v_t+\mathbb{P}(v\cdot\nabla v)\Big)\Big(\eta_t\nabla^2v+v\nabla\eta\nabla^2v+v\nabla^2v\Big)dS\\ &+\int\Big(v_t+\mathbb{P}(v\cdot\nabla v)\Big)\cdot\Big(\nabla(\Delta)^{-1}\Delta(v\cdot\nabla\mathscr{H}(n\cdot T\cdot n))\Big)dV.\end{aligned}\tag{6.32}$$

Integrating (6.32) in-time,

$$\int\int [D_t v, \mathscr{A}]v\cdot(v_t+\mathbb{P}(v\cdot\nabla v))dVdt \lesssim \frac{1}{2}\|v_t+\mathbb{P}(v\cdot\nabla v)\|^2_{L^2_tH^1_x}+|||v|||^4. \quad (6.33)$$

Collecting all terms,

$$\begin{aligned}&\int < v_t+\mathbb{P}(v\cdot\nabla v), v_t+\mathbb{P}(v\cdot\nabla v) > dt\\ \lesssim&\epsilon+\frac{1}{2}\|\nabla\mathscr{H}(\eta-F(\eta))\|^2_{L^2_tH^1_x}+|||v|||^4+\|\eta\|^2_{L^\infty_tH^3_x}|||v|||^2+||v||^2\\ &+|||v|||^2\|\nabla^{\frac{5}{2}}\eta\|^2_{L^2_tH^1_x}.\end{aligned} \quad (6.34)$$

Since $\|\nabla^{\frac{5}{2}}\eta\|^2_{L^2_tH^1_x}\lesssim\|\nabla\mathscr{H}(\eta-F(\eta))\|^2_{L^2_tH^1_x}$, (see Lemma 6.7),

$$\|v_t\|^2_{L^2_tH^1_x}\lesssim\epsilon+|||v|||^4+|||v|||^2+\|\eta\|^2_{L^\infty_tH^3_x}|||v|||^2+|||v|||^2\|v_t\|^2_{L^2_tH^1_x}, \quad (6.35)$$

which completes the proof of Lemma. ☐

Lemma 6.7. $\|\nabla(\eta-F(\eta))\|^2_{L^2_tH^{\frac{1}{2}}_x}\lesssim\|\nabla\mathscr{H}(\eta-F(\eta))\|^2_{L^2_tH^1_x}$

Proof. We will use the Dirichlet-Neumann operator. Suppose $\nabla\mathscr{H}(\eta-F(\eta))\in L^2_tH^1_x$. We set $f=\eta-F(\eta)$. For $i=1,2$,

$$R(\partial_i\mathscr{H}(f))=\partial_i(f)-\partial_i\eta\frac{G(\eta)f+\nabla\eta\cdot\nabla f}{1+|\nabla\eta|^2}\in L^2_tH^{\frac{1}{2}}_x. \quad (6.36)$$

Therefore, by the product rule of fractional derivatives,

$$\|\nabla f\|_{L^2_tH^{\frac{1}{2}}_x}\lesssim\|\eta\|_{L^\infty_tH^3_x}\cdot\|\nabla f\|_{L^2_tH^{\frac{1}{2}}_x}+\|\nabla\mathscr{H}(\eta-F(\eta))\|_{L^2_tH^1_x}. \quad (6.37)$$

Here, we use the fact that $G(\eta)$ is a first order pseudo-differential operator. For its properties, see [8]. By the smallness of $||\eta||_{L^\infty_tH^3_x}$, we finish the proof of Lemma. ☐

We prove Proposition 2.5. By (2.17) and the commutator estimate,

$$|||v|||^2+\|\eta\|^2_{L^\infty_tH^3_x}\lesssim\epsilon+|||v|||^4+\Big|\int(\alpha)dt\Big|. \quad (6.38)$$

We replace $\Big|\int(\alpha)dt\Big|$ in (6.38) by (6.23).

$$\begin{aligned}&|||v|||^2+\|\eta\|^2_{L^\infty_tH^3_x}\\ \lesssim&\epsilon+|||v|||^4+\frac{1}{2}\Big(\|v_t\|_{L^2_tH^1_x}+\|\nabla\mathscr{H}(\eta-F(\eta))\|^2_{L^2_tH^1_x}\Big)\\ &+\|\eta\|^2_{L^\infty_tH^3_x}|||v|||^2+\|\eta\|^2_{L^\infty H^3_x}\|\nabla\mathscr{H}(\eta-F(\eta))\|^2_{L^2_tH^1_x}.\end{aligned} \quad (6.39)$$

We substitute $|||v|||^2$ in (6.39) into (6.24).

$$\begin{aligned}&\|\nabla\mathscr{H}(\eta-F(\eta))\|^2_{L^2_tH^1_x}\lesssim\epsilon+\frac{1}{2}\Big(\|v_t\|_{L^2_tH^1_x}+\|\nabla\mathscr{H}(\eta-F(\eta))\|^2_{L^2_tH^1_x}\Big)\\ &+\|\eta\|^2_{L^\infty_tH^3_x}|||v|||^2+|||v|||^4+\|\eta\|^2_{L^\infty H^3_x}\|\nabla\mathscr{H}(\eta-F(\eta))\|^2_{L^2_tH^1_x}.\end{aligned} \quad (6.40)$$

We substitute $|||v|||^2$ in (6.39) into (6.35).

$$\begin{aligned}&\|v_t\|^2_{L^2_tH^1_x}\lesssim\epsilon+|||v|||^4+\|\eta\|^2_{L^\infty_tH^3_x}|||v|||^2+|||v|||^2\|v_t\|^2_{L^2_tH^1_x}\\ &+\|\eta\|_{L^\infty_tH^3_x}|||v|||^2+\frac{1}{2}\Big(\|v_t\|_{L^2_tH^1_x}+\|\nabla\mathscr{H}(\eta-F(\eta))\|^2_{L^2_tH^1_x}\Big)\\ &+\|\eta\|^2_{L^\infty_tH^3_x}\|\nabla\mathscr{H}(\eta-F(\eta))\|^2_{L^2_tH^1_x}.\end{aligned} \quad (6.41)$$

By adding (6.40) and (6.41),

$$\begin{aligned}&\|v_t\|^2_{L^2_tH^1_x}+\|\nabla\mathscr{H}(\eta-F(\eta))\|^2_{L^2_tH^1_x}\\ \lesssim&\epsilon+|||v|||^4+\|\eta\|_{L^\infty_tH^3_x}|||v|||^2+\|\eta\|^2_{L^\infty_tH^3_x}|||v|||^2\\ &+|||v|||^2\|v_t\|^2_{L^2_tH^1_x}+\|\eta\|^2_{L^\infty H^3_x}\|\nabla\mathscr{H}(\eta-F(\eta))\|^2_{L^2_tH^1_x},\end{aligned}\tag{6.42}$$

which is the end of the proof of Proposition 2.5.

By (6.23) and (6.42), we have the estimate of $\Big|\int(\alpha)dt\Big|$.

$$\begin{aligned}\Big|\int(\alpha)dt\Big|\lesssim&||\eta||^2_{L^\infty_tH^3_x}|||v|||^2+|||v|||^4+|||v|||^2\|v_t\|^2_{L^2_tH^1_x}\\ &+\|\eta\|^2_{L^\infty_tH^3_x}\|\nabla\mathscr{H}(\eta-F(\eta))\|^2_{L^2_tH^1_x}.\end{aligned}\tag{6.43}$$

Therefore, we finish the proof of Proposition 2.4.

Proposition 6.8. *Korn-type inequality*

$$\begin{aligned}\|\nabla\mathscr{A}v\|^2_{L^2_tL^2_x}\lesssim&\int<\mathscr{A}v,\mathscr{A}v>dt+|||v|||^4+\frac{1}{2}\|\nabla\mathscr{A}v\|^2_{L^2_tL^2_x}\\ &+\|\nabla\mathscr{H}(\eta-F(\eta))\|^2_{L^2_tH^1_x}.\end{aligned}\tag{6.44}$$

Proof. Since $\mathscr{A}v$ does not vanish at the bottom, we cannot apply Korn's inequality directly to $\mathscr{A}v$. But, from the momentum equation,

$$\mathscr{A}v+\nabla\mathscr{H}(\eta-F(\eta))-v\cdot\nabla v+\mathbb{P}(v\cdot\nabla v)=-v_t-v\cdot\nabla v.\tag{6.45}$$

Since the right-hand side of (6.45) vanishes at the bottom,

$$\begin{aligned}&\|\partial(\mathscr{A}v+\nabla\mathscr{H}(\eta-F(\eta))-v\cdot\nabla v+\mathbb{P}(v\cdot\nabla v))\|^2_{L^2}\\ \lesssim&\Big\langle\mathscr{A}v+\nabla\mathscr{H}(\eta-F(\eta))+(\mathbb{P}-I)(v\cdot\nabla v),\ \mathscr{A}v+\nabla\mathscr{H}(\eta-F(\eta))\\ &+(\mathbb{P}-I)(v\cdot\nabla v)\Big\rangle.\end{aligned}\tag{6.46}$$

Therefore, we have that

$$\begin{aligned}||\nabla\mathscr{A}v||^2_{L^2L^2}\lesssim&\int<\mathscr{A}v,\mathscr{A}v>dt+|||v|||^4+\frac{1}{2}\|\nabla\mathscr{A}v\|^2_{L^2L^2}\\ &+\|\nabla^2\mathscr{H}(\eta-F(\eta))\|^2_{L^2L^2},\end{aligned}\tag{6.47}$$

which completes the proof of Proposition 2.6. □

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REFERENCES

[1] R. Adams, "Sobolev Spaces," Academic Press, INC, 1975.
[2] D. Ambrose and N. Masmoudi, *Well-posedness of 3D vortex sheets with surface tension*, Commun. Math. Sci, **5** (2007), 391–430.
[3] T. Beale, *The initial value problem for the Navier-Stokes equations with a free surface*, Comm. Pure. Appl. Math, **34** (1981), 359–392.

[4] T. Beale, *Large time regularity of viscous surface waves*, Arch. Rational Mech. Anal, **84** (1983/84), 307–342.

[5] A. Bertozzi and A. Majda, "Vorticity and Incompressible Flow," Cambridge University Press, 2002.

[6] D. Coutand and S. Shkoller, *Unique solvability of the free boundary Navier-Stokes equations with surface tension*, preprint, arXiv:math/0212116.

[7] D. Coutand and S. Shkoller, *Well-posedness of the free-surface incompressible Euler equations with or without surface tension*, J. Amer. Math. Soc, **20** (2007), 829–930.

[8] W. Craig, U. Schanz and C. Sulem, *The modulational regime of three-dimensional water waves and the Davey-Stewartson system*, Ann. Inst. H. Poincar Anal. Non Linaire, **14** (1997), 615–667.

[9] R. Danchin, *Estimates in Besov spaces for transport and transport-diffusion equations with almost Lipschitz coefficients*, Rev. Mat. Iberoamericana, **21** (2005), 863–888.

[10] D. Ebin, *The equations of motion of a perfect fluid with free boundary are not well posed*, Comm. Part. Diff. Eq, **10** (1987), 1175–1201.

[11] D. Lannes, *Well-posedness of the water-waves equations*, J. Amer. Math. Soc, **18** (2005), 605–654.

[12] T. Lundgren and P. Koumoutsakos, *On the generation of vorticity at a free surface*, J. Fluid Mech, **382** (1999), 351–366.

[13] D. Sylvester, *Large time existence of small viscous surface waves without surface tension*, Comm. Partial Differential Equations, **15** (1990), 823–903.

[14] J. Shatah and C. Zeng, *Geometry and a priori estimates for free boundary problems of the Euler equation*, Comm. Pure Appl. Math, **61** (2008), 698–744.

[15] J. Shatah and C. Zeng, *Local wellposedness of interface problem*, preprint.

[16] V. Solonnikov, *Unsteady motion of a finite mass of fluid, bounded by a free surface*, J. Soviet Math, **40** (1988), 672–686.

[17] V. Solonnikov, *Unsteady motion of an isolated volume of a viscous incompressible fluid*, Math. USSR-Izv, **31** (1988), 381–405.

[18] A. Tani and N. Tanaka, *Large-time existence of surface waves in incompressible viscous fluids with or without surface tension*, Arch. Rational Mech. Anal, **30** (1995), 303–314.

[19] R. Temam, "Navier-Stokes Equations - Theory and Numerical Analysis," AMS, 2001.

[20] S. Wu, *Well-posedness in Sobolev spaces of the full water wave problem in 2-D*, Invent. Math, **130** (1997), 39–72.

[21] S. Wu, *Well-posedness in Sobolev spaces of the full water wave problem in 3-D*, J. Amer. Math. Soc, **12** (1999), 445–495.

[22] P. Zhang and Z. Zhang, *On the free boundary problem of three-dimensional incompressible Euler equations*, Comm. Pure Appl. Math, **61** (2008), 877–940.

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